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The thesis examines the theory of lattice models used in statistical physics from a mathematical perspective, and provides a case study of box-crossing property by proving Russo-Seymour-Welsh estimates in percolation model and the box-crossing property in the critical FK Ising (random cluster) model. The thesis is self-contained in its use of probability theory, but a general measure theory background is assumed.

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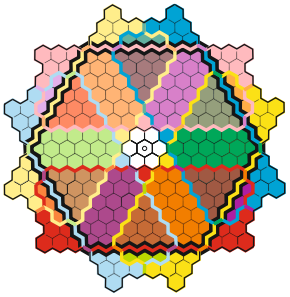

University of Helsinki

Master's thesis

Lattice models of statistical physics and a case study of box-crossing property

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Contents

1. Abstract

The purpose of this thesis is to examine the theory of lattice models of statistical physics in general with emphasis in the mathematical perspective, and to provide a case study of box-crossing property appearing in lattice models by proving so-called Russo-Seymour-Welsh estimates on the so-called percolation model and by proving the box-crossing property of the critical FK Ising (random cluster) model.

2. Introduction

The subject of this paper are the probabilistic lattice models used in physics (similar type of models are used also in other scientific fields, for example even in finance); these models are also termed statistical lattice models in this paper and in wider literature (indeed the term "probabilistic lattice models" is coined by the author to be used only in this paper). This is in opposition to deterministic lattice models, which do not contain probabilistic aspects and of which there are a large number of. We do not however study these deterministic models in this paper. In literature, both probabilistic and deterministic lattice models are referred to simply as lattice models, and in this paper we adopt the term "lattice model" for a probabilistic lattice model.

Lattice models are probability-based models of statistical physics, that are applied to the study of the structure of matter and the study of *phase transition* (so-called *critical phenomena*), like when water changes into ice; besides this, lattice models are also used in the study of many other interacting systems. As the plural suggest, there is not one but many different lattice models. However, the idea of lattice model is common to all different models; all models study a *lattice* (of which there are different kinds of), that is, a graph with a given set of vertexes and edges that is regular in the sense that it can be extended to cover the Euclidean space, usually but not always through recursion of some geometric object (so actually the more mathematically accurate term would be *tessellation*), and in this lattice to some vertexes or edges (depending on the model in question) is assigned in random a state (there can be two or more states). After that assignment the properties of these random state-configurations are studied; what aspects of them are considered interesting depends on the model.

Lattice models have been studied since the early 20th century, the year 1925 being the year that can be considered as a starting point (though some antecedents are easy to find, but these are minor in scientific importance) due to the introduction to the wider scientific community of the most widely studied and in a reasonable sense archetypal model, namely the so-called *Ising model* (modelling a magnetic substance in the atomic level) by *Ernst Ising*; also the solution to the most simple case of the model, namely the one-dimensional case, was published in 1925 (in a scientific journal; the result was presented in 1924 in a thesis) by Ising, making 1925 the year of the publication of the first rigorous and non-trivial important result in the field. Although some progress in the field followed in the years after 1925, most notable being the results of *Rudolf Ernst Peierls* in 1933 that two-dimensional Ising model has a phase transition and the invention of the *Bethe ansatz*, a method for solution of lattice models (applicable only to some lattice models) by *Hans Albrecht Bethe* in 1931, the field did not advance rapidly during the 1920s and 1930s. During 1940s and 1950s new models of importance started to emerge in the field (although the Ising

model has retained its central role to this day), and more momentum in the research of the field started to build up, the most important result of this era being the full solution (under some assumptions) of the 2-dimensional Ising model by *Lars Onsager* in 1944. Especially in and after 1960s the field started to attract increased attention, and while the field had previously been studied mostly by physicist, it started to attract mathematicians too (of course it is impossible to say definitely who of the early researchers of the field was or was not a true physicist or mathematician, this is only the impression the writer has acquired). There has been a lot of interest in the mathematical study of the field in recent two decades, for example, in the last two congresses where the Fields medals have been awarded, one of the recipients has been a researcher of the subject, namely *Wendelin Werner* and *Stanislav Konstantinovich Smirnov*, respectively. Lately much interest has been directed to establishing a connection between lattice models, which are discrete in nature and fields (as studied by physics) which are continuum-type objects through a limit process called *scaling limit* (in which the system is scaled so that the individual constituents of the system become microscopic and only the macroscopic behaviour of the system becomes manifested). The most notable of the unproven conjectures associated with these studies is, that the geometrical structures that present themselves in the lattice models in plane are also present in limiting fields and that these planar geometrical structures are *conformally invariant*, id est, their properties as probabilistic objects do not chance when operated upon by a conformal mapping. The mathematical study of the models focuses on this and other questions of 2-dimensional lattice models; the 3-dimensional lattice models are almost entirely an uncharted territory in sense of exact analysis both mathematically and physically.

This paper does not examine the above question of continuum limit, but rather concerns itself with the discrete lattice models themselves. This study assumes from the reader no previous knowledge of lattice models or even the discrete mathematics used in the study of lattice models, but contains an introduction to the relevant concepts of discrete mathematics, specifically graph theory. However, it is assumed that the reader is familiar with abstract measure theory, and many results concerning measure theory are only stated and not proved, for measure theory is not the main aspect of this study. It is also assumed that the reader is familiar with basic probability theory and mathematical analysis. This study also includes an elementary introduction to the statistical physics behind the study of lattice models and to different lattice models; this introduction is by no means complete, and it does not even begin a closer study of models mentioned. After those introduction sections this study focuses on a group of discrete estimates called the *Russo-Seymour-Welsh estimates* in the particular lattice model of *percolation* (model of a random structure of, say, a porous rock). These estimates show that the percolation model has the so-called *box-crossing*

property. After proving those estimates we prove the box-crossing property of *Fortuin-Kasteleyn random cluster model* (which will be introduced below) in the special, so-called *critical FK Ising-case*. After developing the box-crossing property of the above mentioned models in its own section, we take a quick look of a so-called *arm exponent* (which are a group of estimates in the above mentioned models), namely the *half-plane one-arm exponent* in the last section, to show the reader an application of the box-crossing property. Before all this, however, we begin with a section introducing the reader to various concepts of discrete mathematics used later in the paper.

2.1. **Acknowledgements.** In making this study the writer, *Petri Pyry Tapani Tuisku*, has been guided and assisted by *Antti Heikki Kemppainen*, and the work was approved by *Antti-Jukka Kupiainen*, both from the University of Helsinki. The author has also had enlightening conversations with *Kalle Perttu Juhani Kytölä* and *Clément Olivier Hien Minh Hongler*. *Jacques Herman Henri Perk* was kind enough to merit the writer with a private communication clarifying some aspects of different Potts model derivative models. *Geoffrey Richard Grimmett* and *Hugo Duminil-Copin* gave the writer valuable information about the Russo-Seymour-Welsh estimates in percolation and box-crossing property in random cluster model in a private communication. The titlepage uses LaTeX 2_ε-code from [79]. The figures depicting real simulations (the diagrammatic ones are due to the author) are from *Antti Kemppainen*'s personal collection of figures that he has produced through simulations of his own programming, and from the web pages of *Vincent Beffara* and *Michael John Kozdron*. The figures from Vincent Beffara's web page are based on simulations programmed by him; the figure from Michael Kozdron's web page is based on simulation by *Edward Jon Doolittle*. The main results of this study are based on the article [17]; the influence of the book [38] appears on many occasion.

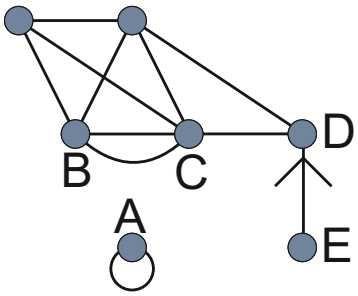


FIGURE 1. A general multigraph. The vertexes are represented as grey balls and edges are lines connecting these balls. The vertex A has loop connected to it; vertexes B and C share multiple edges. Vertexes E and D have a directed edge between them in the direction from E to D. This graph is actually planar, although the figure does not represent a planar embedding of the graph.

3. Introduction to discrete mathematics used in the study of lattice models

It is pursued to use standard notation; however, because the subject is rather novel and has developed remarkably in the last years, the notation used in the study of the subject has not been standardised due to lack of mathematical tradition; also the fact that the subject has been studied both by physicist and mathematicians has lead to varying notational customs. Due to these and other matters, there might be some features in the notation that are typical only to writer himself or to the different materials used as sources.

We use the standard set theoretic notation and the standard notation of mathematical analysis and algebra. Usually we do not denote multiplication explicitly; when we do, we use the symbol $\cdot$. The symbol $*$ is used as a special symbol whose meaning is defined case-by-case; it does not mean complex conjugate, which we denote by a line placed over the variable or number. For a finite set A we denote the number of elements of A as $|A|$; if A is infinite, we denote $|A| = \infty$. For any set A, we denote the power set $\mathcal{P}(A)$.

3.1. **Graph theory.**

As mentioned above, lattice models are probabilistic models on objects called graphs. Here we present the theory of these objects, *graph theory*, in the extent required in this study. The reader that already knows graph theory can omit this subsection as we present only the basic aspects of graph theory.

In *discrete mathematics*, the theory of graphs appears nearly everywhere, and this has lead to graph theory becoming an extremely voluminous theory with applications in nearly all fields of science. Due to the extent of graph

theory it is not possible to present here but a small and elementary fraction of this theory.

The *graph* $G = (V, E)$ is simply a pair of the set of *vertexes* V and *edges* (or *links*, not to be confused with knot-theoretic links) $E \subseteq V^2$. For any graph G, we denote the *set of vertexes* of G by $V(G)$ and similarly the *set of edges* $E(G)$. We assume that the set of vertexes is finite or countable (generally in graph theory this assumption is not needed). The set of edges contains *ordered pairs* (a, b) where $a, b \in V$, which are always called *directed edges*. If the set of edges contains both ordered pairs (a, b) and (b, a), it is said that the graph and the set of edges have an *undirected edge* or simply *edge* between a and b. This pair of directed edges (a, b) and (b, a), that is, the ordered pairs (a, b) and (b, a) jointly, is referred to as an *unordered pair* $\langle a, b\rangle$ or just (*undirected*) *edge* $\langle a, b\rangle$, that is, the elements (a, b) and (b, a) are "permanently joined together" and the result is understood as one element of the edge set $\langle a, b\rangle \in E$ and the operations concerning this one element $\langle a, b\rangle$ then concern both (a, b) and (b, a). The idea of graphs is that the edge set simply describes the relation between the different vertexes of V. Directed relations (a, b) mean that a is in relation to b and not necessary vice versa; it is also said that there is an *arrow* from a to b in graph G. Undirected relation $\langle a, b\rangle$ mean that a is in relation to b and b is also in relation to a and we say that there is an edge between a and b in graph G. We call a and b the *endpoints* of the edge $\langle a, b\rangle$ or directed edge (a, b), and in the case of (a, b), we say that a is the *beginning of the edge* and b is the *end*. Unless we explicitly say otherwise, we always assume that the relations presented by the edges are undirected. In some graph theoretical studies, it is allowed for two vertexes to have multiple edges between them (in this case the definition of an edge is adjusted accordingly) or it is allowed for a vertex to be in relation with itself, in which case we say that there is a *loop* in the vertex. We will assume henceforth that graphs have no multiple edges or loops, also we only consider so-called *simple graphs*.

When there is an edge between vertexes $a, b \in V$, we write $a \sim b$, and we say that a and b are *neighbours*. If there is not an edge between the vertexes, we write $a \nsim b$. A *path* from $x_0 \in V$ to $x_n \in V$, $n \in \mathbb{N}$ in graph G is a finite sequence $x_0; e_1; x_1; e_2; ...; e_n; x_n$ where $x_i \in V$ and $e_i \in E$ and $e_i = (x_{i-1}, x_i)$ for all $1 \leq i \leq n$. We say that the x_i are the *vertexes of the path* and e_i are the *edges of the path*, and we say that the path's *length* is n or that there is *n steps in the path*. If $x_0 = x_n$, the path is a *cycle*. If there is a path in graph G connecting $x, y \in V$, then we write $x \leftrightsquigarrow y$. We see that $\leftrightsquigarrow$ is an equivalence relation in V, and its equivalence classes are called the *connected components* or *clusters* of G. If there is only one connected component in G, then we say that G is *connected*. If we have two graphs $G = (V, E)$ and $H = (W, F)$ such that $W \subseteq V$ and $F \subseteq E$, we say that H is a *subgraph* of G.

By *graph metric* we mean the metric δ in the set of vertexes V defined as the infimum of the lengths of paths connecting two vertexes x, $y \in V$.

By a *sphere with center $y \in V$ and radius $r \in \mathbb{N}$*, $S(y, r)$, in this metric we mean the set $\{x \in V \mid \delta(x, y) = r\}$, and by a *ball with center $y \in V$ and radius $r \in \mathbb{N}$*, $B(y, r)$, in this metric we mean the set $\{x \in V \mid \delta(x, y) \leq r\}$. When the graph we are interested in is presented in the Euclidean plane, we may call a sphere a *circle*.

We shall focus our attention to so-called *planar graphs*, which are graphs that can be embedded in the plane, that is, drawn into the plane in such a way that their edges do not intersect, apart from their endpoints (the vertexes). The representation of such a graph in the plane in such a way that the edges of the graph are drawn as to not intersect is called the *planar embedding of a graph* or just *plane graph*; in general the representation of a graph such that its vertexes are points and its edges are curves in the Euclidean space is called an *embedding* of the graph into Euclidean space. In what follows we usually make a distinction between two different embeddings, even if they represent the same graph; in general graph theory, this distinction is usually not made, and graphs are studied up to isomorphism. This is because in lattice models the geometric properties of graphs are important, whereas in general graph theory they are not, since graphs are studied as themselves, that is, as objects independent of their geometric representation, and changing the embedding of a graph changes the geometry of the graph. The results we will present are (mostly) independent of the embedding and thus depend only on the graph, but because we will use geometrical information in our proofs (in a dominant way), we will consider graphs only with given embeddings in the space.

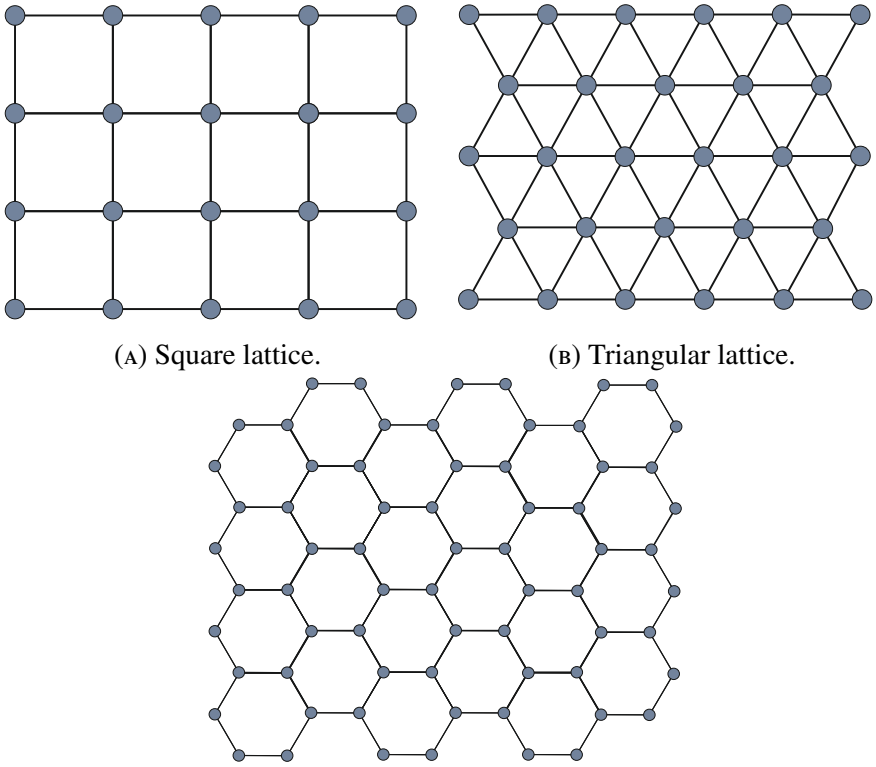
To be even more precise, we study graphs that are called planar *straight-line graphs*, meaning that each edge of the graph is a straight line segment. Since we are interested in some geometric properties of these graphs, it can be said that our interest is directed to the field of *geometric graph theory*. The graphs we study are so-called *geometric periodic graphs*, where by the word geometric we mean to make an opposition to the "graph theoretic" *periodic graphs* that are simply graphs that when operated with some *graph-operator* (an *operator* operating on graphs, that is, a mapping attaching a new graph to a given graph) have a *period*, id est, operating on the graph repeatedly yields ultimately the same graph. Geometric periodic graph is an *Euclidean graph*, that is, a graph embedded into some Euclidean space $\mathbb{R}^n$, which is *periodic*, that is, there exists a basis of that Euclidean space whose corresponding translations induce symmetries of that graph (id est, application of any such translation to the graph embedded in the Euclidean space leaves the graph unchanged) [96]. Three-dimensional geometric periodic graphs are also called *crystal nets*, especially in crystallography and in chemistry.

In the study of lattice models the concept of a *boundary* of a graph manifests itself often. This concept is in general not defined in a standard way but different writers have differing definitions. In graph theory, the following definitions are frequent: the *boundary* of H with respect to G, $\partial_G H$ or

just ∂H, of a subgraph H of G is the set of vertexes of H that in G have a neighbour that is not in H; the *outer boundary* of H with respect to G is the set of vertexes of G that are not in H but have a neighbour in H. In this study and more widely in the study of lattice models one usually takes into consideration the geometrical aspects of the graph, or to be precise, the geometrical aspects of the representation of the graph in Euclidean space, and thus the boundary of the graph in this paper and in the study of lattice models means the portion of the graph (vertexes and edges) that can be geometrically understood to form the "natural boundary" of the representation of the graph.

In mathematics, the term "*lattice*" can refer to an object of graph theory, group theory, Lie theory or order theory; we consider only *lattices* of graph theory, and use the term "lattice" for these objects only (in common English, the term "*lattice*" is also used to refer to the technique of construction of the lattice-like structures we introduce below, or to the objects resembling a lattice-structure as introduced below, so the term might refer to anything from art to furniture to clothing; the important thing is the structure inherent of the object). Usually a "*lattice graph*" in a sense that is followed in physics and in mathematics literature is a graph that is formed when the Euclidean space is covered through repeating some geometric shape, with the vertexes of the graph being the vertexes of those shapes and the edges of the lattice being the edges of those shapes, so actually the more accurate term is *tessellation graph* instead of lattice graph, and the whole field should be called "tessellation models" (note: in geometry, a three-dimensional tessellation is sometimes called "*honeycomb*"; this should not be mixed with the two-dimensional honeycomb lattice). We do not however follow this terminology, nor does any other writer.

There are many technical definitions of a lattice graph that vary slightly, but the "archetypal" definition could be something as follows: a *lattice graph* $\mathbb{L}$ is a graph whose set of vertexes V is a discrete additive subgroup of $\mathbb{R}^d$, where by *discrete* we mean that there is a neighbourhood of the origin that contains only one lattice point, the origin ([58], page 14) and in which the nearest neighbours (in the Euclidean metric) are connected by an edge. This or other definitions are not helpful in practical calculations, nor are they followed to the letter in physics (or even in mathematics) literature, so we just define the lattices we shall be using. The most common lattices in use are the d-dimensional cubic lattice $\mathbb{L}^d$ and the triangular and hexagonal lattices. *d-dimensional cubic lattice* is the lattice that has as the vertex set $\mathbb{Z}^d$ and which has an edge between any two vertexes that are Euclidean distance 1 apart. *Triangular lattice* is the plane lattice formed by covering the plane with equilateral triangles. The vertexes and edges of the triangles are the vertexes and edges of the lattice. *Hexagonal lattice* is constructed similarly. Note that according to the above technical definition, the hexagonal lattice is not a lattice at all; but the name "hexagonal lattice" is well



(A) Square lattice.

(B) Triangular lattice.

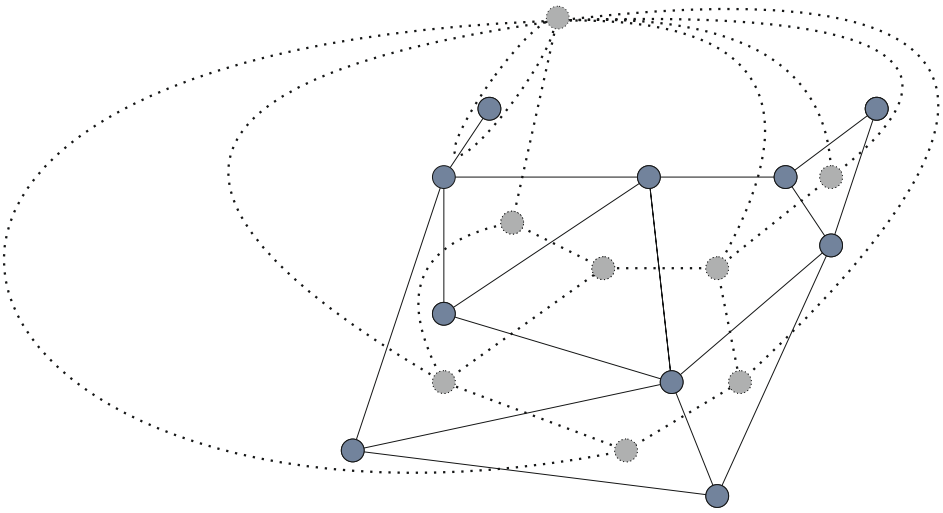
(C) Hexagonal lattice.

FIGURE 2. Finite sections of the most used lattices (square, triangular, hexagonal).

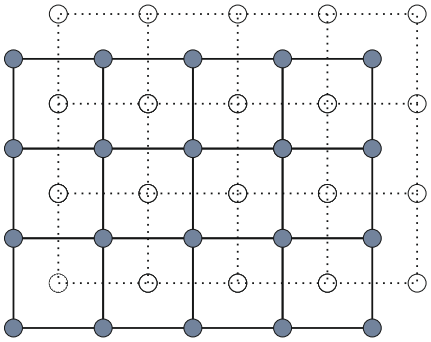
established, so we shall use it. Many other lattices exists, although many of them, like the hexagonal lattice, are actually not lattices at all.

3.2. **Planar duality.** In graph theory, for every planar graph G we define the *dual graph* (also called *Whitney dual graph*) G_d as follows. The vertexes of the dual graph are the faces of the original graph (including the infinite face, if any), and there is an edge of the dual graph between two vertexes (not necessarily different two) of the dual graph if and only if there is an edge in the original graph that is intersected by this dual edge; that is, each edge of the original graph is intersected by exactly one dual edge that connects those vertexes that correspond to the faces of the original graph separated by the edge of the original graph (note that it is possible that the faces separated by the original edge actually are the same face, in which case there is a loop in the dual graph). Note that this definition might lead to the dual graph having loops and being a *multigraph*, that is, a graph that has multiple edges between two vertexes or multiple loops from one vertex. However, in situations where we apply duality, this does not happen.

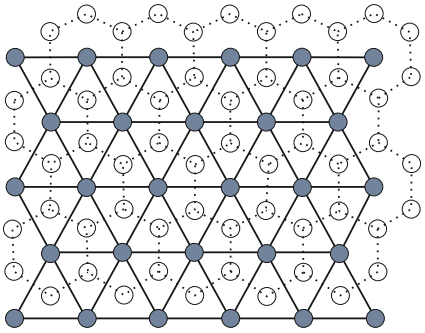
We see that the *square lattice*, that is, 2-dimensional cubic lattice, is self-dual and that triangular and hexagonal lattices are dual lattices.



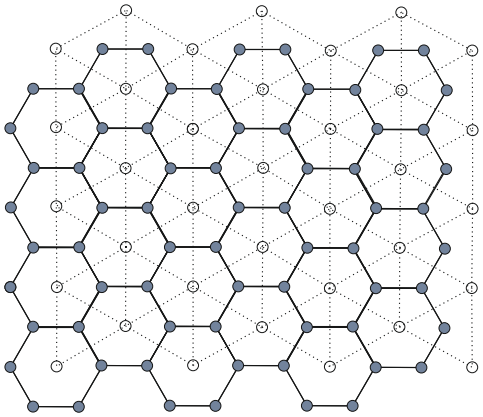
(A) The dual graph of a planar graph.



(B) The dual of the square lattice.



(C) The dual of the triangular lattice.



(D) The dual of the hexagonal lattice.

FIGURE 3. The dual graph of a general planar graph and finite sections of the dual lattices of the most used lattices (square, triangular, hexagonal).

Usually in the context of studying lattice models, when one speaks of a *dual graph* of graph that is a subgraph of some original lattice, he does not actually mean a dual graph in the above graph theoretic sense, but instead some "dual-like" object defined on the dual lattice. In this paper we also utilize these "dual-like objects" which we shall define more rigorously in a case-by-case approach.

3.3. **Euler's formula.** In graph theory, the well-known *Euler's formula*, named after *Leonhard Paul Euler* (since Euler, many generalizations of the formula have been made and usually all of those generalizations are called "*Euler's formula*") who formulated it around 1750 for convex polyhedra (but did not prove it; first to prove it was *Adrien-Marie Legendre* in 1794 [22]) is the following statement:

Theorem 3.1. *For a connected finite planar graph $G = (V, E)$, the number of vertexes $|V|$, the number of edges $|E|$ and the number of faces F (including the infinite face of a finite planar graph) satisfy*

$$|V| - |E| + F = 2.$$

We will prove a special case where the graph is simple, id est, has no loops or multiple edges.

Proof. We advance by induction on the number of edges. If the graph has zero edges, then since it is connected, it is just an isolated vertex and $|V| = 1 = F$ (remember to count in the infinite face), so we have $|V| - |E| + F = 2$. If the graph has one edge, since it is connected, it must be just a graph of two vertexes joined by an edge, hence $|V| = 2$, $|E| = 1 = F$, so we have $|V| - |E| + F = 2$. If the claim holds for a graph having n edges, then let us study a connected finite planar simple graph that has $n + 1$ edges. Choose an edge e and, if deleting it will not make the resulting graph disconnected, delete it. The edge bounded two faces (if it would bound only one face, it would mean that the other vertex connected to the edge could be encircled so that one could start and end at the edge and not cross other edges during the encircling - see figure 4 - but then deleting the edge makes the graph disconnected), so deleting it reduced the number of edges and faces by one. The number of vertexes remained the same; hence the sum $|V| - |E| + F$ remained the same. If on the other hand deleting the chosen edge makes the graph disconnected, then it must be the case that the component to which another one of the endvertexes of the edge belongs to can be "isolated" by making a loop-path in the plane around that component such that the loop does not intersect any edges of the graph; see figure 4. This means that contracting the edge does not result in loops or multiple edges, id est, the graph remains simple and connected in the process of contracting the edge. However it is possible that the contraction of the edge makes the graph non-planar. This can be avoided by rescaling the component which could be made isolated so that it becomes so small that "it can be seen in the direction of the contracted edge without obstacles", and after this rescaling,

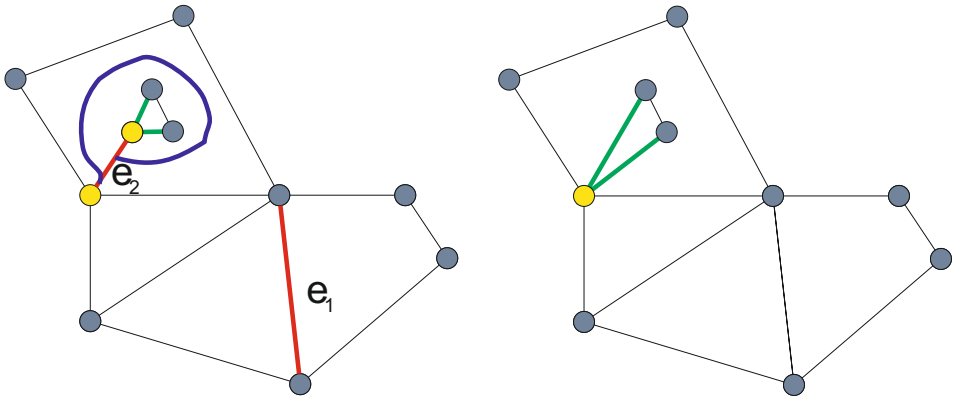


FIGURE 4. The proof of Euler's formula. On the left hand side, the edge e_1 can be deleted without making the graph disconnected whereas the edge e_2 cannot; however, the component to which one of the vertexes of e_2 (yellow) belongs to can be "isolated" by encircling it (blue) such that the encircling loop does not intersect any edges of the graph. Then the edge e_2 can be contracted as on the right hand side.

the contraction of the edge does not lead to non-planarity. Note that we need not make the edges of the graph to some general curves but we can assume the edges are straight lines. Note also that the rescaling does not effect the number of vertexes, edges or faces, so we can think of the rescaling as just a technical auxiliary device. So contract the edge; this results to the number of edges and vertexes being diminished by one and the number of faces staying constant, so again the sum $|V| - |E| + F$ stays invariant.

But now we have a connected finite planar simple graph that has n edges; by induction hypothesis, the sum $|V| - |E| + F$ for this graph equals 2; since the sum was invariant in the changes we made, the sum equals 2 also in the original graph of $n + 1$ edges. By induction, we have proven the claim (this proof is from [21]). □

Next we generalize the above Euler's formula to general (simple) finite planar graphs (not necessarily connected). The resulting formula is also called *Euler's formula*.

Theorem 3.2. *Let $G = (V, E)$ be a finite simple planar graph and let $|V|$, $|E|$, F, C be the number of vertexes, edges, faces (including the infinite face) and connected components of it. Then we have:*

$$|V| - |E| + F - C = 1.$$

Proof. Let the connected components of G be $G_1, \ldots, G_C$; let $|V_i|$, $|E_i|$, F_i be the number of vertexes, edges and faces (including the infinite face) of connected component i, $i = 1, \ldots, C$. Now using the above Euler's formula

for each connected component separately, we can write the equation

$$\sum_{i=1}^{C} |V_i| - |E_i| + F_i = \sum_{i=1}^{C} 2 = 2C$$

which we can manipulate as follows:

$$\begin{aligned}\sum_{i=1}^{C} |V_i| - |E_i| + F_i &= \sum_{i=1}^{C} |V_i| - \sum_{i=1}^{C} |E_i| + \sum_{i=1}^{C} F_i \\ &= |V| - |E| + \sum_{i=1}^{C} F_i \\ &= |V| - |E| + (F + (C-1)) \\ &= |V| - |E| + F + C - 1 = 2C\end{aligned}$$

where we have noted $\sum_{i=1}^{C} F_i = F + (C-1)$, which stems from the fact that the faces of the connected components of the graph are also faces of the whole graph, but because the infinite face is counted in the sum once for each connected component, the infinite face appears in the sum C times, but it should appear only once; hence the sum "overshoots" the number of faces F by $C - 1$. From the above equation we can solve

$$|V| - |E| + F - C = 1. \tag{3.1}$$

□

The reason for presenting Euler's formula is that we shall need it in one part of the argument in proving the box-crossing property below. Besides that, Euler's formula is also used in many occasion in the study of two-dimensional lattice models, and it is a useful peace of information in that sense also; and it is one of the "canonical" famous results of graph theory, and it is good to know also in that respect.

4. Statistical physics

As this study is concerned with a subject in the field of mathematical physics that is applied to statistical physics, it is perhaps in order to have a little exposition of statistical physics. The following material is standard and may be found in any textbook on the subject; we have used as a reference a textbook used in teaching of the Master's-level courses "Statistical physics I" and "Statistical physics II" in the University of Helsinki Department of Physics [2].

Statistical physics (or *statistical mechanics*, a more old-fashioned term) is a field of physics that studies systems that have many interacting independent objects, many meaning in this case something in the order of 10^{23}. That is, the number of independent objects in the systems studied in statistical physics is so large that solving the equations of motion for each object (although classical and quantum mechanics allow one to formulate these equations) is practically impossible; therefore in order to gain information about the behaviour of the system, only the "medial" aspects of the dynamics of the objects in the system are taken into consideration by imposing certain probability measures on the space of possible microscopic states of the system, that is, the space of microscopic states of the system that correspond to the observed macroscopic state of the system. This is the main idea of statistical physics. By the terms "microscopic/macroscopic state of the system" used above we mean the following: a *macroscopic state* of the system refers to the system being characterized by macroscopic quantities such as temperature, pressure and volume. The *microscopic state* of the system refers to the complete characterization of the system by fixing all the *degrees of freedom*, that is, quantities that can evolve in the system; for example in the study of gases all the degrees of freedom can be fixed by giving the coordinates and momentums of each molecule of gas (an idealized gas molecule is a mathematical point having only three properties: mass, velocity and location) in the large system of gas studied. For example, two systems of gas can have the same macroscopic parameters of temperature, pressure and volume, but different microscopic parameters in the sense that the molecules of gas that make up those systems have different velocities and spatial coordinates in the two systems. Further "states of the system" include the *thermodynamic equilibrium state* of a system meaning that all the macroscopic thermal quantities of the system remain constant. *Thermodynamic non-equilibrium state* means that the thermodynamic quantities alone cannot describe the system.

There are two formulations of statistical physics: the *classical statistical physics* and the *quantum statistical physic*s, and to both of these one can

attach relativity so that one could for example have *relativistic classical statistical physics*. The relativistic nature of the problem is taken into consideration in the correct formulation of the systems *energy function* or *Hamiltonian* (defined below) in both classical and quantum statistical physics, and by requiring that the time-evolution of the system follows relativity.

In a more general setting in physics, the statistical physics gives a way to derive macroscopic properties of systems from their microscopic structure, thus forming a microscopic basis to the field of physics called *thermodynamics*, that studies the processes of energy transportation (in the form of for example *work* and *heat*) between systems independently of their microscopical properties (it is said that thermodynamics is a phenomenological theory, basing its analysis on laws of nature derived from a few key observations rather than a fixed set of axioms; note that a "*law of nature*" in physics is simply a property of nature that cannot be derived from other properties within the theory used in analysis). Let us note that thermodynamics concerns itself only with thermal equilibrium states of the system, in which the macroscopic properties of the system remain unchanged. Therefore the concept of time can be fractioned out from the analysis of systems, id est, thermodynamics (although its name) does not concern itself with the dynamical aspects of the system in the mechanical sense, but studies only the changes of energy of the system (also the *thermal dynamics*). Note that although one usually associates thermodynamics with studying the processes related to heat and temperature change, thermodynamics as a theory does not concern itself with the precise method of the transfer of energy in systems that it analyses; therefore, thermodynamics can be used to analyse a variety of processes, for example the behaviour of magnetic materials, in which the system consist of a lump of magnetic material whose magnetization under external magnetic field is studied.

4.1. **Quantum mechanics.** This is a very short introduction to quantum mechanics for the purposes of illuminating the quantum statistical physics we are about to present. A previous familiarity with the subject is assumed, and by no means should this presentation be considered exhaustive.

In *quantum mechanics* [100], the microscopic state of the system is given by a single vector in the quantum mechanical state space. The *quantum mechanical state space* of a single object is the product space of the complex Hilbert-spaces of the quantities related to that object; different quantities have different state spaces, for example for position- and momentum-quantities, the appropriate Hilbert space is some function space whose elements are so-called *wave functions*; the magnetic spin quantity's state space is the $\mathbb{C} \times \mathbb{C}$-vector space [106]. The *quantum mechanical state space* of a system of multiple objects is the product of state spaces of individual objects of the system; the nature of these objects can assert certain further conditions on the states of the whole system. For example, if the "objects" are elementary particles, they can be either *bosons* or *fermions* and

these two particle species have a different collective behaviour because two bosons can be in the same quantum state but two fermions cannot (indeed the bosons and fermions demand a different analysis in statistical physics; boson and fermions obey different distributions, the so-called *Bose-Einstein* and *Fermi-Dirac statistics* respectively). In case the number of objects in the system (for example, the number of elementary particles) is not constant, the correct state space for the system is a so-called *Fock space*, that is, a direct sum of the state spaces of n-object systems for all n.

In quantum mechanics in general, a *measurable quantity* of the system under measurement corresponds to a self-adjoint operator on the state space of the system. The values of measurable quantities need not be deterministic but they can be random, and one obtains the distributions of these quantities by expressing the state vector of the system in the basis formed from the eigenvectors (because the quantum mechanical state space is a linear space, normal functional analysis can be applied) of the appropriate operator (the operators are self-adjoint so their eigenvectors span the space), and then calculating the inner product of the state vector of the system and the vector of the Hilbert space that one obtains by operating with the appropriate operator to the state vector of the system. The probability to observe an eigenvalue of the appropriate operator as the value of measurement of the corresponding quantity is the modulus squared of the coefficient of the eigenvector corresponding to the eigenvalue in the above linear combination. The *expected value* of measurement is the value of the above inner product. The calculation of an inner product of vectors φ and ψ is usually done by evaluating the function of the dual vector space corresponding to the inner product with φ at ψ. All the physical properties that a system has are encoded by the state vector, and all the measurable quantities can be extracted from this vector through similar process as described above.

An operator of special importance is the *Hamiltonian operator*, which corresponds to the total energy of the system. The eigenvalues of this operator are called *energy eigenvalues* or *orbitals* in atomic context, and the corresponding eigenvectors are called *energy eigenstates*. An important fact is that these eigenvectors form a basis in the Hilbert space; the further development of quantum theory usually utilizes (only) this basis. The Hamiltonian operator also dictates the time-evolution of the quantum state vector through the *Schrödinger equation*.

In quantum mechanics, a system whose quantum state is completely know is said to have *pure state* (or to be in a pure state) and a system whose state is unknown in the sense that one cannot specify a single state vector for the system but must instead rely on the statistical analysis described above in the general statistical physics-section is said to have a *mixed state* (or to be in a mixed state). Note that there is no quantum mechanical uncertainty involved in the distinction between a pure and a mixed state but the difference stems from our incompetence as observers to find out the state of the system in terms of a unique state vector. Quantum mechanically all

system are in a pure state all the time, but it is the difference in our analysis of the system that makes the mixed state appear. This is completely similar to the situation in classical statistical physics; given any classical physical system, all the microscopic quantities of the system are uniquely defined all the time (for example, in a volume of gas each molecule has a definite location and velocity all the time and the evolution of those quantities is even deterministic), but since we cannot in practice find out the values of these quantities, we choose to study the distribution of these quantities over a large set of similar systems instead, thus adding randomness into the picture not as a property of the system under study but as a property of our analysis of the system. In quantum mechanics the only difference is, that a system whose state we can uniquely determine still has randomness in itself in the sense that all the measurable quantities of the system are not deterministically dictated, only their distributions are (the time-evolution of the distributions being given by the Schrödinger equation is also deterministically dictated) but the values of the quantities are as random as the distributions allow them to be. Thus there are two levels of randomness and probabilities in the statistical analysis of a quantum mechanical system; the level of the randomness that is inert of the quantum mechanical system, and the level of randomness that stems from the statistical analysis of the system.

4.2. **Statistical physics.** As said above, in statistical physics the central idea is to put a measure in the set of possible states of the system, the *state space* (the state space is also called *phase space* in classical setting). The procedure of introducing this measure is called *ensemble theory* in physics. Simply put, the ensemble is the collection of systems having different microstates but common macrostate. Hence the ensemble can be seen as the space of possible microscopic states of a system given its macroscopic state. The measure in the state space is a measure in the ensemble. Let us note now in passing that if the concept of time is included in statistical physical analysis, it enters through the above measure being time-dependent and the time-dependency being such that it obeys the *Hamiltonian equations* of classical mechanics in the case of classical statistical physics or the *Schrödinger equation* time-evolution in the case of quantum statistical physics, or the relativistic versions of the above equations in the relativistic case.

In the operator formalism of quantum mechanics the above measure on the state space is formalized by a so-called (time-dependent) *density operator*, and the notion of integration with respect to measure is formalized as taking traces of (the possibly infinite) matrixes of operators multiplied by this density operator (operator multiplied by density operator is yet another operator, which can be expressed as an infinite dimensional matrix in the basis of the quantum mechanical Hilbert state space; the trace of this matrix is the one described above). However, mathematically this formalization

gives nothing new; basically the Hilbert-state space of a quantum mechanical system is given a probability measure that describes the weight of each basis vector of the space, and hence the weight of each vector. The density operator of quantum mechanics must satisfy certain conditions and obey the *von Neumann equation* (of motion of the density operator).

There are basically three different type of measures in the state space that are considered in statistical physics: the microcanonical, the canonical and the grand canonical ensembles (remember that a mathematician can consider the word ensemble as a synonym for measure). The *microcanonical ensemble* is the simplest. In the classical case, microcanonical ensemble assigns to every state in the state space (or some subset of it) having a fixed energy E a uniform weight. If there is only a finite number of states possible, the uniform weight is simply $\frac{1}{N}$ where N is the number of states; in case there is an infinite number of states, one of course cannot have a concept of uniform weight unless one describes a measure with respect to which the weight is uniform; usually in physics, the measure is a scaled Lebesgue measure in some Euclidean space or subdomain of it in cases where the states are such that they can be described as n-tuples of the reals. There is also a slight variation of the microcanonical ensemble in which the energy is not fixed to an exact value but it is fixed that the energy is between E and $E + \Delta E$, in which both E and ΔE are parameters of the ensemble. Otherwise this "new type" of microcanonical ensemble is similar to the "old one" (these "new" and "old"-terms are coined by author to be used here only), that is, a uniform measure.

In quantum mechanics, there is no point in saying that a general state has a definite energy E because the quantum randomness of measurable quantities. Only energy eigenstates have a fixed energy, and quantum mechanical microcanonical ensemble studies these eigenstates. Let us note that in quantum mechanical case, the spectrum of the Hamiltonian operator (the operator corresponding to energy) of any system is always discrete if the volume of the system is finite, regardless if the number of quantum mechanical objects in the system. Hence there is only a finite number of energy eigenstates below any energy value E.

One way of defining the *quantum microcanonical ensemble* of states with energy E would be to define it so that gives the states with energy E the weight $\frac{1}{N}$, where N is the number of energy eigenstates having energy E (energy eigenvalues can be degenerate, meaning the multiple eigenstates can correspond to the same energy); and other states weight zero. If $N = 0$ above, the density operator would be the zero-operator (that does not fulfil the normalizing requirement for the density operator, namely the trace of zero-operator is not one). However more usually in quantum mechanical setting, the microcanonical ensemble is defined such that it is effectively the indicator function of whether the state has energy that is in the interval $[E, E + \Delta E]$ (where E and ΔE are given as parameters to the microcanonical ensemble) normalized by the number of energy eigenstates in the energy

interval (a finite number) [92]. Also the "finite uniform" variation of microcanonical ensemble, id est, the case in which there is only a finite number N of states under study and each state has weight $\frac{1}{N}$ appears in quantum mechanics in specific applications and instances.

It can be shown that the microcanonical measure (classical and quantum) maximizes the so-called *entropy* of the system - defined as $S = -\operatorname{Tr}\rho\ln(\rho)$ where Tr means trace (with respect to some basis of the Hilbert space) and ρ is the density operator in the quantum setting (note that taking the logarithm of an operator can be done); in the classical continuum setting the entropy is $S = -\int_\Gamma p(P)\ln(p(P))dP$ where $p(P)$ is the probability density function and dP is a measure on the state space Γ (usually a scaled Lebesgue measure) - if the energy, volume and number of particles of the system is given (that is, only those states that have given energy, volume and number of particles are considered in the state space).

The *canonical ensemble* has the measure that we shall call "thermodynamic measure" below, since we shall mostly focus on it. The measure is defined as follows: in the classical setting the measure of the state x with energy $H(x)$ (the function announcing the energy being the Hamiltonian function) in temperature T is taken to be $\frac{1}{Z}e^{-\frac{1}{k_BT}H(x)}dx$ where Z is the normalizing constant, that is, the measure of the whole state space and k_B is a constant relating the temperature to energy - it turns out that measuring temperature is the same as measuring the internal energy (that is, energy related to the interactions that are internal to the system) of the objects that constitute the system in the microscopical level - the *Boltzmann constant* $k_B = 1.3807 \times 10^{-23}\frac{J}{K}$. The term dx refers to the measure for the states; in classical statistical physics, the states can usually be represented as n-tuples of reals and dx is then the Lebesgue measure in suitable subspace of some Euclidean space. In a quantum mechanical setting, we again focus our attention into the density operator that is defined by the above formula such that the energy function is replaced by the Hamiltonian operator and the measure dx is forgotten; then the formula gives us a proper operator. The normalizing constant is chosen such that the trace of the operator is one. Canonical ensemble is the one that maximizes the entropy when the expectation of energy is a given constant (note: expectation, not energy itself) and the number of particles and volume are also constant. Note that the classical setting and the quantum setting can be related by interchanging integration and taking of trace and functions with operators [81]; this is a more general principle.

The *grand canonical ensemble* is the measure that maximizes the entropy under assumption that the expectation of the energy and expectation of the number of particles in a system are given (again note expectation, not exact value) and the volume is constant. The measure is $\frac{1}{Z}e^{-\frac{1}{k_BT}(H(x)-\mu N(x))}dx$ where Z, k_B, T, $H(x)$ are defined as above and μ is a constant, so-called *chemical potential* that basically tells the amount of *free energy* (that is, energy that is extractable from the system without the system losing temperature or

volume) is possessed by individual particles [82] and $N(x)$ tells the number of particles in state x. In quantum mechanics, the corresponding density operator is formed similarly as in the canonical ensembles case (the $N(x)$ is replaced by the *number operator*) [87].

4.3. **Phase transition.** The above subsection dealt with the field of statistical physics. Now we move onward to study the field of thermodynamics. As said above, statistical physics forms the microscopic basis of thermodynamics, and when applying statistical physics, the goal is usually to be able to say something about the thermodynamical properties of the system based on the statistical physics analysis (in particular this is the motivation for the study of lattice models in physics). In the thermodynamical analysis of the system, perhaps the most important quantity is the *free energy function* or (*thermodynamic*) *potential* which means the amount of energy the system can give to its surroundings without the system losing temperature or volume [90]; there are also different types of *free energies* or *thermodynamic potentials* than the above described *Helmholtz free energy*, for example the *enthalpy* or the *Gibbs free energy*, but the point of all these thermodynamic potentials is to announce the amount of energy that can be extracted from the system without the system losing some thermodynamic quantity. The free energies are functions of the thermodynamical quantities (these quantities can be identified as real parameters) describing the macroscopic state of the system (which can be identified with the n-tuple of real parameters, the real parameters taking the values of the thermodynamical quantities describing that macroscopic state). In the space of possible macroscopic states (by which we mean the set of possible values of the thermodynamical quantities the system can attain, endowed with the structure inherited from an Euclidean space) of the system, the *macroscopic state space* (term coined by the author), the regions where this free energy function no longer is a smooth function of its parameters are called regions of *phase transition*. In a point of the macroscopic state space where the free energy is not a smooth function, but one can approach the point such that it is smooth along the path of approach in macroscopic state space, it is said that a *phase transition* occurs and the point is called a *critical point*; the *order of the phase transition* depends on the order of the non-smoothness of the free energy. Phase transition is called first or second order; *first order phase transition* is such that the first (partial) derivative of the free energy is discontinuous; in a *second order phase transition* the first (partial) derivative is continuous and the second (partial) derivative has a discontinuity. The free energy is always continuous in a phase transition. Phase transitions can also be classified as *continuous* or *discontinuous*, depending on how the so-called *order parameter* (see below) behaves at phase transition (whether it is continuous or discontinuous).

Phase transition happens at specific temperature that depends on density, pressure and other properties. Below and above the transition temperature

the free energies or thermodynamic potentials of the system have different analytic behaviour; this leads to the *equation of state* (that describes the dependency between different thermodynamic state variables of the system) to differ on different sides of the transition temperature. This change of the equation of state manifests itself in the sometimes quite drastic change in the behaviour of the system. Let us note that mathematically, a finite system cannot experience a mathematically sharp singularity (that is, the singularities in the partial derivatives of the thermodynamic potentials are not pure singularities but instead they are "extreme behaviour", meaning that the values of the function may fluctuate quite wildly but no real singularity is present). Mathematical singularity can only be achieved at the *thermodynamic limit* where $V \to \infty \leftarrow N$ such that $\frac{N}{V} = \text{constant}$.

For a second order (or continuous) phase transition it is typical that as one enters the lower temperature phase, some symmetry of the system is *spontaneously broken*. For example in *ferromagnetic phase transition* (a magnetic transition that happens when magnetic substance that is placed in an external magnetic field is cooled) the broken symmetry is the rotational symmetry of the spins, id est, the spins start to favour some directions over others. In a *superconductive phase transition* of electron systems or in the *superfluidity phase transition* of helium, the so-called *gauge symmetry* is broken. In certain phase transitions of crystalline structure materials, the lattice of the crystal is broken. The second order phase transition in some cases can be modelled by so-called *Landau theory* (developed by *Lev Davidovich Landau* and partially generalized by *Vitaly Lazarevich Ginzburg* and hence called also *Ginzburg-Landau theory*; its is applied especially in the study of superconductivity but it applies to other phase change phenomena too), that describes the behaviour of the system near the critical point of temperature.

The phase change of statistical systems can also be studied through *mean field theories*. The main idea of mean field theory is to simplify any model under study by neglecting the individual interactions between the objects of the model and instead define an effective field that only takes into consideration the mean properties of the interactions; for example, in the Ising model (that we present in greater detail below) one can forget about the interaction of spins of the individual atoms and instead argue that each spin, due to being in symmetric position with respect to the rest of the spins, feels the effect of the other spins in a similar way; this effect can then be (again on average) taken into account by adding a constant magnetic field into the model with which each spin then interacts.

Let us also mention that the spatial dimension of the model has large relevance to the behaviour of the model under phase transition; if the dimension is greater than the (*high*) *critical dimension* (usually 4), the model is accurately described by mean field theory. On the other hand, when the dimension is lower than the *low critical dimension* (usually 2 or 1 depending on the model), no phase change at all can take place.

4.4. **Critical phenomena.** As said above, it is typical of a second-order phase transition that some sort of spontaneous symmetry breakage takes place in the transition point. This symmetry breakage can be quantized by introducing an *order parameter*, which can be a real number or vector of many real valued quantities which has value 0 in the symmetric phase and other than 0 in the asymmetric phase. What quantity (quantities) should be considered as the order parameter depends on the model studied; for example, in ferromagnetic systems the order parameter is the *magnetization* of the system, id est, the net difference between spins up and down. The order parameter must be chosen correctly to match the system under study.

Near the critical point, the system usually starts to have *critical fluctuations*, meaning that the system starts to have long range interactions, even though the physical interactions taking place inside the system are only of short range. At criticality, the system can exhibit *infinite-range interactions*; for example the value of the spin of some atom affects the spin of another atom no matter how far away the other atom is. This kind of system looks similar in all scales (above the scale of distances between the atoms); it is *scale invariant* and *self-similar*.

When a system is brought close to critical point, the system becoming scale invariant causes many different *critical phenomena* to appear in the system. The critical phenomena and indeed the phase transition in general are not yet fully understood neither physically nor mathematically. However, physicist have been able to develop some theories of these phenomena; let us now introduce some of these theories that physicist consider to be "quite right" (indeed, some might even consider some of the theories below as "standard").

In the critical point, the non-analyticity of the free energy leads to many thermodynamic variables being singular. There has been proposed a *scaling hypothesis* saying that all the singular parts of all thermodynamic variables scale as a power of *correlation length* (id est, the length at which two objects can have effect on each other) near the critical point. The exponent of the above power relation is called the *critical exponent* (*critical index*). The exponent depends of course on thermodynamic quantity studied (and let us remark that the critical exponents are sensitive towards the dimensionality of the model); exponents relating to certain quantities have specific names. One can heuristically argue (one could also use the *Widom scaling theory*, which bases itself on the *Widom hypothesis*: the singular part of the free energies near the critical point are so-called *generalized homogeneous functions*) certain relations, so-called *scaling laws*, between these critical exponents. Let us also mention just by name the *Kadanoff* (*Leo Philip Kadanoff* 1966) *block transformation* and *renormalization group* (*RG*) theory; these theories are used in the further study of criticality to derive the critical exponents from the underlying microscopic theory.

4.5. **Areas of research.** Next we shall only briefly present some questions that statistical physics presents to mathematicians in the study of lattice models. Our list is of course not decisive but we have tried to give at least some kind of picture of the big questions in the field.

- In statistical physics one proves that all interesting thermodynamic properties of the system, especially the free energy can be calculated from the so-called *partition function*, also called *zustandssumme* or *normalizing constant* $Z = Z(\textit{Thermodynamic variables})$, which is just the measure of the whole (microscopic) phase space (according to the above introduced microcanonical, canonical or grand canonical measure, depending on which measure is appropriate in the analysis of the system) by derivation and other easy operations. Therefore the most important quantity to discover is $Z = Z(T)$, that is, the zustandssumme as a function of temperature. This is why calculating the zustandssumme has such a central place in statistical physics that if a model's zustandssumme can be calculated as an explicit function of temperature, it is said that the model is *solved* or is *solvable* or *integrable*. From a mathematicians viewpoint this might seem odd; after all, finding out the total mass of the distribution gives mathematician only quite trivial information about the distribution, so a mathematician would not consider a model solved after being able to calculate the total mass. However, I hope the above clarifies to the reader why physicists do. Hence one of the main goals of the mathematical study of models of statistical physics, and lattice models in particular, is to calculate the zustandssumme Z, allowing one to study the phase transition through calculation of the free energy from the zustandssumme.
- In statistical physics one is also interested in *correlation functions* of different quantities, that is, functions $C(x, y) = \langle xy\rangle - \langle x\rangle\langle y\rangle$ where we denote the correlating quantities with x, y and the expectation with respect to the appropriate ensemble measure with $\langle\rangle$. Especially interesting is the correlation length that one can deduce from the correlation functions, due to its relation to the critical exponents.
- The mathematically rigorous study of critical exponents, scaling laws and other critical phenomena is still very much a work in progress. Also the above mentioned renormalization group and related theories lack mathematical foundations.
- A mathematically open problem of importance is the hypothesis of *universality* (in continuous phase transition). The hypothesis is that all models that have the same spatial dimension and whose order parameter have the same dimension (the *dimension* of the order parameter is the number of its independent components) have the same critical exponents, correlation functions (up to constant factor), free energy function (up to constant factor and proper normalization of units) and scaling relations, id est, the behaviour near the critical

point is the same. Hence should this hypothesis be true, only the behaviour of the simplest possible model needs to be studied since the behaviour of all models (that have the same dimensions) is the same. Only the critical temperature depends explicitly on the model in question.

- A famous big open question in the field, as told already in the introduction section, is the question of *conformal invariance,* that is, whether the models exhibit it. In two dimensions, this question has been the object of much mathematical interest in the last decade.
- We feel that as a last remark, it is in order to point out that quite little is known mathematically rigorously about the different lattice models of statistical physics. There does not exist a mathematical solution to most of the models addressed here and below. Also models that have realistic dimensions have proven to be mathematically quite intractable at least for now. In general one could say that the one-dimensional models are mathematically solved, the two-dimensional models are under study and nobody (mathematician or physicist) has a clue about the three-dimensional models.

5. Introduction to lattice models

This section aspires to give a general and qualitative introduction to different lattice models; although there are many different lattice models, the basic structure of these models can be seen to be analogous; especially when considering the probabilistic lattice models of statistical physics, the Ising model, to be introduced below, can be considered as the archetypal lattice model (with some exceptions). Much of the content of this section is from the books [38], [34], [74] and from the Wikipedia Internet encyclopedia.

5.1. **Definition of lattice model.** There is no clear definition of what is and what is not a *lattice model*. However, the main idea of lattice models is to study the different aspects, both local and global in the scale of the system under study, of interaction between multiple different agents by representing the possible interactions with the help of a lattice graph. This is true whether the lattice model is probabilistic or deterministic in nature and we shall try and follow this criteria as our definition of lattice model: a *lattice model* is a model in which a lattice graph is used to model interactions between multiple agents. The phenomena that arises because of the interactions of the systems agents are called *cooperative phenomena*. Note that we do not specify the area of study in which lattice models are applied, nor do we specify the agents or objects whose interactions (the nature of which we also do not specify) are under study; we choose not to limit these matters since we accept that lattice models are used widely in the scientific field and we shall focus instead on the structure of the model. As said above, we shall however direct our attention to a subfield of lattice models, namely *probabilistic lattice models*, in which some random aspect is added; either into the interactions between the objects in the lattice, or as is the case in this paper, to the configurations of the objects that are placed in the lattice. And as stated above, we furthermore limit ourselves to models of interest in statistical physics.

Let us also mention here that we do not consider time evolution in any form when we study our probabilistic lattice models. This might seem to the reader a bit strange choice, since obviously time in the physical context is a major issue, allowing one to speak about equilibrium, convergence, metastability and even chaos, concepts that clearly should be dealt with in the macroscopic description of the matter that bases itself on the microscopic properties of matter, such as lattice models in statistical physics. The needed theory to couple time with lattice models has been developed utilizing *stochastic processes*, for example (*Monte Carlo*) *Markov chains* and so-called *Glauber process* (or *Glauber dynamics*; special cases of the above are called *Gibbs sampler* or *heat-bath algorithm*) and *Swendsen-Wang dynamics*, the latter two being a special kind of Markov chains developed specifically for use with lattice models (for example in a Glauber process the single edges or vertexes to which some state is assigned in the model

chance their state - so-called *spin-flip process* - at such a rates that the equilibrium measure is the measure of the model), but this is a whole another industry of its own and surprisingly quite independent from the general study of lattice models both in mathematics and in physics; in order to keep the present study somewhat contained, we do not delve into the subject more (for random cluster model, and also in an introductory manner for other models, the subject is presented in [36], chapter 8). The idea of neglecting time evolution in a lattice model is that the lattice model represents the equilibrium of the system and is therefore time-independent (or at least the fluctuations around the equilibrium are small).

5.2. **Non-probabilistic lattice models.** There are lots of lattice models being used in different fields of science that do not contain any probabilistic aspects. These models can be found for example in computer science where lattices are used as computational tools and theoretical ancillaries (to avoid confusion, let us note that the *lattice model of information flow* in computer science is order-theoretic [59], not graph-theoretic) and in discrete mathematics and in the fields applying it. Let us note that in discrete mathematics and the fields applying it, for example in computer science, the terminology is a little different from what we use and the models that can be considered as lattice models are studied in these fields in the theory of general graphs and networks (*network* is a special type of graph, yet sometimes the term *network* is also used to mean a general graph) and these models might not be called lattice models; for example the *theory of diffusion* in networks/graphs is quite closely related to percolation (see below) and some of the other lattice models introduced below. Also in physics, in the fields of material science, thermodynamics, solid state and condensed state physics and in the theory of mechanics of interacting particle systems there are many of these deterministic lattice models, such as the *Frenkel-Kontorova* (*FK*) *model* of classical mechanics, or different *mass-spring-damper models* of vibrations. These type of models are not discussed in this paper. We will only focus on probabilistic lattice models which we will refer to as *lattice models* from here on.

5.3. **Random configurations on a lattice.** Lattice models are probability models defined on lattices; that is, models that attach random configurations of some objects to lattices. The study of these models should be considered as part of *probability theory*, that itself is a rather independent part of *measure theory* and *mathematical analysis*. As noted above we assume the reader has knowledge of measure theory and basic probability theory, and we will not review these subjects here.

There is no standard technical definition of a "*lattice model*" in literature; different writers give slightly varying definitions and most writers do not even attempt to define "lattice model" in generality but they resort to defining the individual models of interest to them separately. But since the structure of many lattice models is remarkably similar, we attempt here to

describe something that could be considered as the "general lattice model" for the purpose of introducing the reader to the theoretical background that the majority of different lattice models share. Let us note that there is no clear separation between lattice models and non-lattice models, and indeed the field of lattice models is so diverse that not all the models can share the same theoretical structure, but we have tried to resolve these questions and present a reasonably general theoretical framework here. We shall not prove all our claims below as this is not the aim of this paper.

Technically one might define a *lattice model* naively as follows: when we are given a lattice or in general, a graph $G = (V, E)$, a *lattice model* is a triple (S, Ω, P) of the set of objects S that are assigned to graph G, the configuration space Ω that tells how the objects of S are assigned to the graph G, and the probability measure P in the configuration space Ω (because the σ-algebra we want to examine is always constructed the same way, as is explained below, we need not to specify it). The quadruple (G, S, Ω, P) is called a *realization of the lattice model*, or sometimes lazily, just *lattice model*.

More technically, let $G = (V, E)$ be a graph with no restrictions; we will mention specifically if the model needs to impose some conditions to graph G in order to be well-defined. Let $(S, \mathcal{S})$ be a given measurable space of possible values ($\mathcal{S}$ is a σ-algebra of S); what objects S consist of depends on the model considered. The *space of configurations* or the *probability space* is the product space $\Omega = S^I$, where the *index set* I is typically (that is, in most cases but not always) either the set of vertexes $I = V$, in which case it is said that the model is of *vertex-type* (but not a *vertex model*, a name given by physicist to some models that are actually of edge-type), or the set of edges $I = E$, in which case the model is said to be of *edge-type*. The σ-algebra used is the *cylinder σ-algebra* C of Ω generated by the (finite dimensional) cylinder sets. *Cylinder sets* or *cylinder events* $C \subseteq \Omega$ are sets such that there exist a finite $F \subseteq I$ and a set $A \subseteq S^F$ such that $A = \prod_{i \in F} A_i$, $A_i \in \mathcal{S}$ for all $i \in F$, such that $C = A \times S^{I \setminus F}$ where I is the indexing set. Note that if S is finite and $\mathcal{S} = \mathcal{P}(S)$ (the case used in this paper), A can be any set. Now remember that the *Borel σ-algebra* $\mathcal{B}$ is the σ-algebra generated by open sets, and note that in the special case where S is finite and I countable, when we view the space $\Omega = S^I$ as a product topological space of the topological space S (S has discrete topology) we see through the definition of product topology that $\mathcal{B} = \mathcal{F}$, since the sets that form the basis of the topology and generate $\mathcal{B}$ are cylinders, and there is only a countable amount of these sets, so every open set is in the cylinder σ-algebra, so $\mathcal{B} \subset \mathcal{F}$, and because every cylinder set is open, $\mathcal{F} \subset \mathcal{B}$ (every cylinder set is a union of open sets of the form $B \times S^{I \setminus F}$, where $B = \{\omega\}$ where $\omega \in A$, and F and A are like in the definition of the cylinder set). We note that in the case that S is discrete, each cylinder is both open and closed (since the complement of a cylinder can be expressed as union of cylinders $B \times S^{I \setminus F}$, where $B = \{\omega\}$, $\omega \in S^F \setminus A$, where F, A are as before). Note that

these topological facts hinge on the discreteness and finiteness of S, and in the case S is infinite but measurable space with a Borel σ-algebra it is not true that the cylinder σ-algebra and the Borel σ-algebra are the same in product space, and this makes the analysis of these cases a lot harder.

Last is defined the probability measure P in $\mathcal{F}$ (sometimes called *configuration probability*); it differs from model to model, but there are three types of measures in general:

- *Uniform measure* (microcanonical ensemble): a set of objects of interest, $T \subseteq \Omega$ is defined (what the "objects of interest" are depends on the model) and each configuration $\omega \in T$ is assigned an equal probability, and configurations not in T are assigned probability 0. Note that for this definition to work, the set T must be finite. Usually this demand forces the underlying graph G to be finite, and this is why the uniform measure is usually preliminarily defined only when the underlying graph G is finite. Sometimes a continuous uniform measure is introduced; this only means that a measure is given that is being utilized "as is" in the measurement of configurations (usually this measurement involves integration of the constant function 1 over some subdomains of the configuration space).
- *Product measure*: in the product space $\Omega = S^I$, where I is countable, the *product probability measure* is a measure defined by a countable number of probability measures μ_i, $i \in I$ of the "base-space" S such that for all sets of the form $A = \prod_{i \in J} S_i \times S^{I \setminus J}$, where $J \subseteq I$ is a finite set and $S_i \subseteq S$ are measurable, the value of the product measure $P = \prod_{i \in I} \mu_i$ is $P(A) = \prod_{i \in J} \mu_i(S_i)$. The product measure is defined on the product σ-algebra of the product space, that is, σ-algebra generated by the sets of the form A above, which can be seen to equal $\mathcal{F}$. We do not prove the existence or uniqueness of the product measure, although both properties are valid. Note however that the product measure is a priori defined also on infinite graphs G and does not need to be extended like the measures of other types.
- "*Thermodynamic measure*" (canonical ensemble): the "thermodynamic measure" (the term is not in common use) is essentially the canonical ensemble measure of statistical physics and thermodynamics. The thermodynamical measure inherits many different names form its "physical background", including for example: "*Boltzmann measure* (or *distribution*)", "*Gibbs measure* (or *distribution*)" and "*canonical measure* (or *distribution*)".

  We define the thermodynamic measure of a configuration $\omega \in \Omega$ in the discrete setting, id est, when S is finite and the underlying graph G is finite as

$$P_\beta(\omega) = \frac{1}{Z_\beta} e^{-\beta H(\omega)} \text{ for all } \omega \in \Omega,$$

where $H : \Omega \to \mathbb{R}$ is the *Hamiltonian* (*function*) or *configuration energy*, $\beta > 0$ is a constant (in physics, it is the inverse of temperature T, perhaps scaled by the Boltzmann constant k) and Z_β is the *normalizing constant* or the *partition function* (German: *Zustandssumme*):

$$Z_\beta = \sum_{\omega \in \Omega} e^{-\beta H(\omega)}.$$

The exponent in the thermodynamic measure of a configuration is sometimes called "*Boltzmann factor*" or "*Boltzmann weight*" of the configuration.

To form the sum present in the definition of Z_β, we must demand that the configuration space $\Omega = S^I$ is finite. This is always the case when the space of possible values S and the index set I (which usually is the set of vertexes or the set of edges of the graph G) are finite. This is why the thermodynamic measure is preliminarily defined only on finite graphs G with finite set S, and when we want to study the measure in infinite lattices, we have to extend the measure.

There is also a way to define the thermodynamic measure in the case that S is an infinite but measurable space. Then one defines of finite graph G (and hence for a finite index set I) the thermodynamic probability measure P so that for every measurable (in the product σ-algebra) set $A \subset \Omega = S^I$:

$$P_\beta(A) = \frac{1}{Z_\beta} \int_A e^{-\beta H(\omega)} d\omega,$$

where the constant Z_β is the normalizing constant $Z_\beta = \int_\Omega e^{-\beta H(\omega)} d\omega$ and the measure $d\omega$ is the product measure (or in more generality, some measure on Ω such that Z_β is finite) of the measures in the space S, that is, if $(S, \mathcal{S}, \mu)$ is the measurable space, then $d\omega = \prod_{i \in I} d\mu$; note that since G is a finite graph, so is the index set I and hence the product measure does exist by simple measure theory in the product σ-algebra $\prod_{i \in I} \mathcal{S}$, which in the case (that we assume) that $\mathcal{S}$ is the Borel σ-algebra in S is the same as the Borel σ-algebra in $\Omega = S^I$ (a finite product) and the same as the cylinder-generated σ-algebra in Ω. Note that we have implicitly assumed that the total mass Z_β is finite.

We have not given the definition of the Hamiltonian due to the fact that it is model-dependent; the only condition we impose on the Hamiltonian is measurability. Note also the minus-sign in the definition of the probability measure and the partition function. We could rid ourselves of this sign; however, it is common practice in physics to include the sign and we follow this practice.

As noted above, there is a problem with the fact that some lattice models are defined in a way (id est, the probability measure is defined) that only

makes sense when the underlying graph is finite, yet we usually want to extend the definition into the case where the underlying graph is infinite (usually the lattice needs to cover the whole plane, for example); this process of extending the measure is called taking the *thermodynamic limit* and the resulting measure is called *infinite volume measure*. The way to do this is to define a sequence of measures of the model in question in finite graphs that increase in the sense that each graph in the sequence contains the previous graphs and the sequence in some sense "converges" to the infinite lattice (one possible definition of a lattice could be "a graph in which the above described process is possible and sensible to execute"), and then take the so-called *weak limit* of these measures to find the measure appropriate for the infinite graph.

In general probability theory, a sequence of measures μ_n *converges weakly* to μ, all measures defined on the Borel σ-algebra $\mathcal{B}$, written $\mu_n \Rightarrow \mu$, if $\int_\Omega f d\mu_n \to \int_\Omega f d\mu$ as $n \to \infty$ for all bounded continuous functions $f : \Omega \to \mathbb{R}$. This is equivalent to $\limsup_{n\to\infty} \mu_n(A) \leq \mu(A)$ for all closed $A \in \mathcal{B}$ and also to $\liminf_{n\to\infty} \mu_n(A) \geq \mu(A)$ for all open $A \in \mathcal{B}$. In the case S is discrete, since the cylinders C are both open and closed and they generate $C = \mathcal{B}$, it is enough for weak convergence that $\lim_{n\to\infty} \mu_n(C) = \mu(C)$ for all cylinders C. Conversely, since the cylinders generate the σ-algebra, the condition $\lim_{n\to\infty} \mu_n(C) = \mu(C)$ on all cylinders C defines a probability measure μ that is the weak limit of μ_n.

We describe here naively one way on how to define an infinite lattice measure on the d-dimensional cubic lattice $\mathbb{L}^d$. This definition leads to a limiting measure that is usually called the *free measure* since no conditions are imposed on the boundaries ("boundary" being interpreted appropriately) of the graphs that are used as the base for the sequence of measures whose weak limit is studied. If some kind of conditions are imposed on the boundary (for example, if the model assigns some values to the vertexes, it could be demanded that all the vertexes on the boundary of the graphs have the same assigned value), other measures are found. In the case of the random cluster model, which we shall study more carefully below, one other important example of infinite volume measure is the *wired measure*, which is the weak limit in the case that the "boundary" vertexes of the graphs that are used as the base for the sequence of measures are connected pairwise by additional edges added to the graphs (so-called boundary condition edges, see the subsection on random cluster model below).

Let us assume that the set of possible values S is a discrete compact metric space, id est, finite (as it will be on many models), and that the index set I is the countable set of edges or vertexes of $\mathbb{L}^d$; the reason we make this assumption is that by the famous *Tikhonov theorem* [109] (*Andrei Nikolaevich Tikhonov*, also transliterated *Tychonoff* or *Tychonov*) $\Omega = S^I$ is also a compact metric space and the cylinders are open and closed in the product topology. More complicated situations demand a more sophisticated approach and we will not go to this here (see [56]).

The idea is to define the measures of the model we are interested about in "*n-boxes*" $\Lambda(n)$, $n \in \mathbb{N}$, which are subgraphs of $\mathbb{L}^d$ such that the set of vertexes of $\Lambda(n)$ is $\{-n, \ldots, n\}^d$ and every edge of $\mathbb{L}^d$ that has points of the above mentioned set as endpoints is also an edge of $\Lambda(n)$. Since $\Lambda(n)$ is a finite graph, we can define the measures of all models on it. We see that in an obvious sense, when $n \to \infty$, then $\Lambda(n) \to \mathbb{L}^d$. Now let us denote the measure on $\Lambda(n)$ by μ_n. We want to define the measure on $\mathbb{L}^d$ to be the weak limit of the μ_n as $n \to \infty$. We do this by considering the μ_n to be probability measures in the space $(\Omega, \mathcal{F})$ where $\Omega = S^I$ and $\mathcal{F} = \mathcal{B}$ is the cylinder-generated σ-algebra.

To be precise, this kind of an interpretation for the μ_n requires the domain of the μ_n to be redefined (the measure μ_n is technically defined on space $\Omega_n = S^{I(\Lambda(n))}$, not on "the infinite space" $\Omega = S^I$) but this can be done by giving the cylinders C of Ω the measure

$$\mu_n(\{\omega \in S^{I(\Lambda(n))} \mid \exists \omega' \in A_C : \omega(e) = \omega'(e) \text{ for } e \in F \cap I(\Lambda(n))\}),$$

where the sets F, A are like in the definition of a cylinder (note that since F is finite, the limits $\lim_{n\to\infty} \mu_n(C) = \mu(C)$ exist). This enlarges the μ_n-measures to domain Ω since the σ-algebra is generated by cylinders. Then one can apply the weak convergence-technique.

Now since S is compact, so is $\Omega = S^I$ for any I by Tikhonov theorem, and hence any infinite family of probability measures on Ω is therefore *tight* (meaning that for any $\epsilon > 0$, there is a compact $K_\epsilon \subset \Omega$ so that $\mu(K_\epsilon) > 1 - \epsilon$ for all measures μ of the family [108]). By Prokhorov's theorem [99] (*Yuri Vasilyevich Prokhorov*) this means that any infinite sequence of measures of the family contains a weakly convergent subsequence [83], [105]. Showing that this limit measure of the subsequence is indeed the limit of the whole sequence is a matter that must be dealt with separately on every model (and indeed has not been solved in all cases, [36], section 5.8).

It has not been yet emphasised but it is not clear that the infinite-volume limit would be unique, id est, independent of the subgraphs ($\Lambda(n)$ above) used in its definition. In fact this is not always the case; the finer analysis of this phenomenon is however sidestepped here, for little can be said that would be general to all lattice models (in the case of the random cluster model, the main model of this paper, the interested reader should turn to [36]).

All in all, our intuition is that for example the thermodynamic measure in the infinite volume is somehow proportional to the Hamiltonian used in its definition; this is true in a sense, but as we are about to see, most of the Hamiltonians we like to consider are not well-defined in the infinite lattice, so our intuition cannot be literally true.

The reason of our interest in the infinite lattice measures over the finite lattice measures stems from the fact that there is no phase transition in the finite case, and after all, phase transition and related critical phenomena are the reason for our interest in lattice models (this may be an overstatement

but not by much). One can see that in a finite case no phase transition occurs; for if our underlying graph is finite, the space of configurations (usually) is too; hence every probability is simply a sum over the probability weights of configurations of the appropriate event, and since the weights of configurations usually are smooth functions of the parameters (notably temperature) then these sums are also smooth; hence no mathematical singularities and no phase transition can occur in the finite case.

5.3.1. *Geometric properties of random configurations on a lattice.* Above we have described lattice models in quite general setting, and we have tried to describe which kind of questions about lattice models are studied in general. This study in particular, when we study the percolation and random cluster models (to be introduced below) in more detail below, is directed towards a subfield of lattice model studies called *discrete random geometry* or just *random geometry* when one studies such geometric objects that the discreteness does not play a major role. In this subfield one is interested in some random geometric objects imposed on the lattice-graph by the random configurations. Let us now introduce some terminology about the random geometric objects appearing on lattice models.

Let $\omega \in \Omega$ be a configuration of the lattice model. Now ω imposes on graph G (or to be more precise, on the embedding of the graph on Euclidean space) many geometric properties. One of the important objects is the constant-value path from x to y, $x, y \in V$ (which technically is not only a geometric but also a graph-theoretic object), which is simply a path in the graph G such that each vertex or edge traversed is assigned the same value in ω. If this values is $s \in S$, we can speak of a *s-path*. A *s-cluster* is a set of vertexes that can be connected to each other with s-paths. When the underlying graph G is presented in the Euclidean space and it is appropriate, we also define many other geometry-related shapes on the graph, and we might refer to geometry-related terminology like "inside of some shape" (the rectangle-shape will be heavily used below); these terms will be used in a context that makes their meaning obvious.

5.4. **Uniform spanning tree, Uniform forest, Uniform connected subgraph.** The models named above are presented here because they and references to them appear frequently in the literature concerning lattice models; however, it is debatable whether they themselves can be considered to be lattice models or whether they should simply be considered graph-theoretical objects with probabilistic aspects, especially since they do contain very limited interactive properties (many graph theoretical models can be "made to look like" lattice models by simply considering them on a lattice graph). Besides these models, there are many other random tree models in graph theory; for more information, see [102].

A *tree* is a graph that is connected and contains no cycles. A *spanning tree* of a graph G is a tree which has the same vertex-set as G. *Uniform spanning tree-model* (abbreviated *UST*) in a finite, connected graph $G = (V, E)$

has the set of possible values $S = \{0, 1\}$ and the configuration space $\Omega = S^E$ and the set of objects of interest T the set of spanning trees of G, when each configuration $\omega \in \Omega$ is interpreted as the subgraph of G that has vertex-set V and edge-set $\{e \in E \,|\, \omega(e) = 1\}$, and the measure is the uniform measure in the set of spanning trees, id est, each configuration that corresponds to a spanning tree has probability $\frac{1}{|T|}$ to occur and the other configurations have probability 0; clearly since G is finite, there are a finite number of spanning trees, and the measure is well-defined. We would like to define the UST model for countable graphs also, but as mentioned above, we see that our definition of the measure makes no sense for infinite graphs. Here we must use the technique of weak limit as described above to extend UST to infinite graphs. Sometimes the resulting limit measure is called *uniform spanning forest* (*USF*) due to the fact that it gives non-zero probabilities to graphs that are not trees. This measure is very different than the uniform forest measure discussed below. Let us also mention that in the case of the UST measure, the free/wired infinite measures are different and are both considered as interesting objects. The free measure is called *free* (*uniform*) *spanning forest* or *FSF* (or *FUSF*), and the other measure is called *wired* (*uniform*) *spanning forest* or *WSF* (or *WUSF*) [43], [61].

Forest is a graph that has no cycles. A *spanning forest* of graph G is a subgraph of G that has the same set of vertexes as G and is a forest. Now the *uniform forest-model* (*UF*) is the model that we get when we assign to each spanning forest of a graph G a uniform probability, completely analogously to the UST model. The *uniform connected subgraph-model* (abbreviated *UCS* and sometimes *USC*) is the model that we get when we assign to each spanning connected subgraph (again, a subgraph that has the same set of vertexes as the original graph) a uniform probability. As the UST-model, the UF- and UCS-models are defined directly only in finite graphs and need to be extended if we wish to study them on infinite graphs. We do not go deeper into this subject.

Uniform spanning tree-model has relevance to the study of electrical networks (both random and non-random) that was recognized by *Gustav Robert Kirchhoff* in 1847 (see [62], page 21, subchapter 1.7, page 22, page 38 and remark on page 111). UST is the subject of [62] chapter 4, and subchapter 4.2 is devoted to the connection of UST and electrical networks. The book [62] also presents the UST, UF, USC-models much more thoroughly than is possible here. Electrical networks are also studied in [62] chapter 9. The chapter 10 of [62] shows that uniform forest-measure is also connected to electrical circuits.

UST, UF, UCS of course have interest also as themselves, and they also have applications to probability theory and graph theory, and to different areas of computer science as well as to the study of other models described in this paper.

5.5. **Random walk, Loop-erased random walk, Self-avoiding walk, Self-avoiding polygon.** The models presented in this subsection are really not lattice models since they do not fill the definition of lattice model via interaction of multiple agents, but as was the case in the previous subsection, the relevance (to this study, that is) of these graph-theoretical probabilistic models stems from the fact that they are frequently referred to in the context of the study of lattice models, and the study of these models in lattice graphs has been a major project in the study of lattice(-like) models.

A *random walk* (abbreviated *RW*) W on a graph G is a random process (a *Markov chain*) that is defined as follows: the process starts at some vertex usually named the *origin*. After that, the process takes steps in the graph along the edges so that each step is random. The process can be a *discrete time random walk*, denoted $(W_n)_{n\in\mathbb{N}}$, in which case each step takes the same amount, say, one unit, of time (represented by the subscript index), so in technical terms, we have that $W_n \in V$ and $(W_n; W_{n+1}) \in E$ for all $n \in \mathbb{N}$, and each transition is random. The other possibility is that the process is *continuous time random walk*, $(W'_t)_{t\geq 0}$, in which case the steps do not all take the same "time", but each step takes a random time to occur. Technically the duration of the steps of the random walk is described by a Poisson process $(N_t)_{t\geq 0}$ (a "*Poisson clock*"), where the positive real parameter t is interpreted as "time", and the continuous time random walk is actually a composition of a Poisson process and a discrete time random walk: $W'_t = W_{N_t}$. What this means is that the random walker walks randomly from one vertex of the underlying graph to a neighbouring vertex such that the time when the random walker "pops up" at the neighbouring vertex is given by the Poisson process (which increases its value by one at random times t). Note that the walker is actually a "jumper": it stays on one vertex and the instantaneously jumps to a neighbouring vertex, without ever begin "on the way", traversing the edge between vertexes. There are also many more different variations of random walk, and we do not attempt to make any kind of exhaustive presentation here. The random walk-model is one of the most studied models of probability, and this study forms a whole theory of its own, which we do not even try to discuss here. Note that the random walk-model does not conform to any of the three types of probability-models that we introduced earlier, except in some degenerate cases, like when the length of a random walk is fixed n, and the random walk steps in every direction with a uniform probability that does not depend on the vertexes encountered by the walk (the so-called *simple random walk*, *SRW*), in which case the random walk can be seen as a uniform graph path of length n.

The *loop-erased random walk* (*LERW*) is a variation of the random walk in which the cycles of random walk are erased immediately after they form in the walk. LERW has connections to the self-avoiding walk discussed below, and LERW is a concept that appears in connection to lattice models, but like the random walk, it also is not a lattice model in the sense explained

above, so we do not discuss it further. If the reader needs more information on random walk or LERW, we refer to [58].

A *self-avoiding walk* (*SAW*) is simply a random graph path that does not intersect itself, id est, never comes back to a vertex it has already visited. We can study self-avoiding walks by defining a probability measure on the space of non-self-intersecting paths of G. Usually the SAW-model has the uniform probability measure as one usually studies SAWs of fixed length $n \in \mathbb{N}$, of which there are only a finite amount of in any lattice (or on "nice" graphs) or on studies SAWs on a finite graph in which case there is only a finite amount of SAWs; therefore, we can give each path a uniform probability. If we want to extend to an infinite graph, we must again use weak convergence. A *self avoiding polygon* (*SAP*) is simply a self avoiding walk that is closed, id est, the first and last vertex are the same. The model of self-avoiding polygons is similar to the self-avoiding walk model. Mathematically, little is known about the SAWs or SAPs (in dimension less than 5), although physicist have provided many conjectures through numerical simulations.

The theory of random walks on a graph has relevance to the theory of electrical networks (seen as a theory of networks and flows) and vice versa (see [62] chapter 2, chapter 3, chapter 9). The random walk has been studied in physics as means to study chain-like polymers and diffusion (and indeed, the *Brownian motion* of small particle suspended in fluid, being shoved around by the pushes of atoms under thermal motion) since the 1950s [107]. Self-avoiding walks were introduced by the Nobel price-winning chemist *Paul John Flory* in the middle of 1930s to model behaviour of solvents and chain-like polymers [104], [94], [85]. SAWs also have earned a place in combinatorial probability theory and statistical physics [70]. SAPs are used in the studies of topology of chain-like molecules. Let us also just mention that SAWs and SAPs have connections to knot (and link) theory.

5.6. **Percolation.** *Percolation*, also called the *random maze* (*model*) (see [34], preface) in older literature, was conceived as a mathematical theory in the late 1950s (as a curiosity let us mention that *Vincent Beffara* mentions in [6], chapter I, of a 19th century reference to a problem that nowadays would be recognised as percolation) by *Simon R. Broadbent* who in 1954 discussed in a preliminary fashion about the percolation question [41], [34] and by *John Michael Hammersley* who published with Broadbent an article in 1957 defining the percolation-model [95]. Let us also mention that the above mentioned *Paul Flory* had also discussed something like percolation in his 1941 article [26], and a special type of percolation called *Erdős-Rényi* (*ER*) *random graph* (*model*) $G(n, p)$ (not to be confused with another random graph model $G(n, M)$, which chooses a graph uniformly from the set of graphs having n vertexes and M edges, that is also called *Erdős-Rényi random graph* (*model*)), that is simply bond percolation (see below) with probability p on the complete graph with n vertexes (a *complete graph* is a

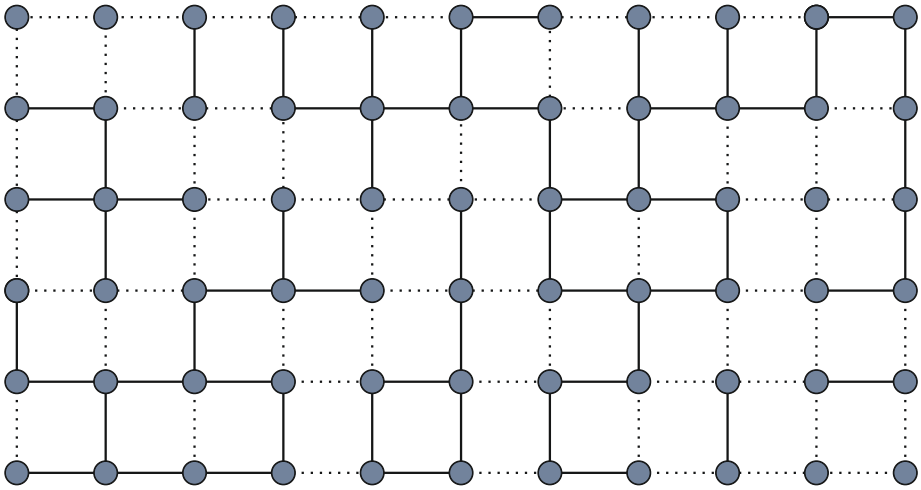


FIGURE 5. A configuration of the bond percolation model on a rectangular subgraph of the square lattice in two dimensions (a diagrammatic figure). Open edges are represented by lines and closer edges are represented as dotted lines.

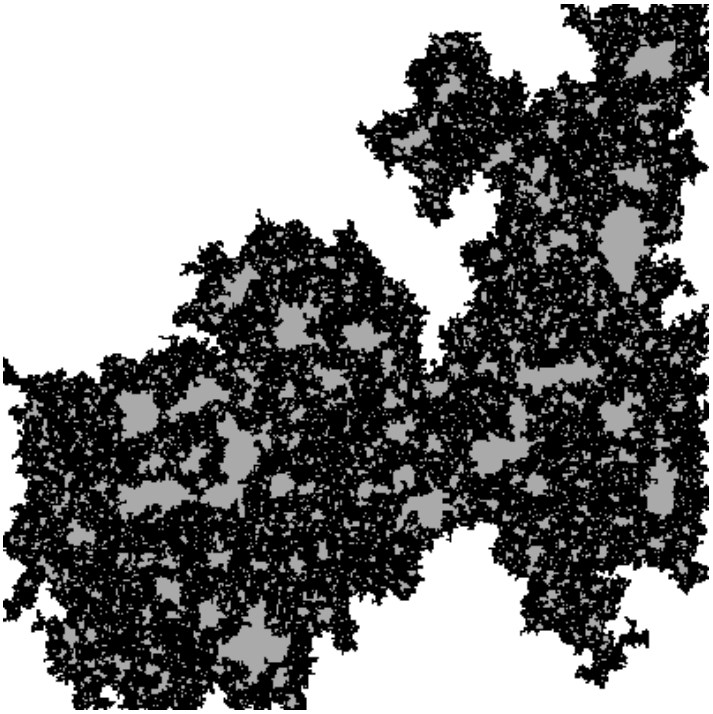


FIGURE 6. A simulation of the bond percolation cluster in the square lattice at criticality. Figure credit: *Vincent Beffara*.

graph in which all the vertexes share an edge) was defined by *Edgar Nelson Gilbert* in 1959 and named after *Paul Erdős* and *Alfréd Rényi*. G(n, p)-model has been studied to great extent in graph theory and discrete mathematics. There are also other "*random graph*" models in graph theory and discrete mathematics that resemble or indeed are some special type of percolation; one finds reference to these *random graph models* every now and then in the lattice model literature. We do not however go further into this matter, as our main focus in this paper are the lattice models and not the graph-theoretic models; for more information see [101], [9].

Although much research has been done on the subject, and many central questions have been answered, there still remain many important unanswered questions. The following exposition is largely based on [34] and [10].

Percolation theory's main idea is that we have a graph (or in general a multigraph), and that the configurations of interest are formed by naming some of the vertexes (also called *sites* or in older texts *atoms*), or edges (also called *bonds* or sometimes *links*), *open* (sometimes called "*active*", "*passable*" or "*undammed*" in reference to edges or "*occupied*" in reference to sites, in older literature) and others *closed* (respectively, "*passive*", "*blocked*" or "*dammed*" edges or "*vacant*" sites) such that each vertex or edge is declared open with probability $0 \leq p \leq 1$ and closed otherwise, independently of the other vertexes or edges (this simplest version of percolation is sometimes called *Bernoulli percolation*). The variant of percolation that attach the open/closed attributes to vertexes is called *vertex* or *site percolation*, and the variant attaching these attributes to edges is called *edge* or *bond percolation*.

More technically, let the underlying, arbitrary graph be $G = (V, E)$ (of interest is the case that G is at least connected), and take as set $S = \{0, 1\}$ and take as the configuration space $\Omega = S^I$, where I is the set of vertexes or edges of G, depending on whether we want to study site or bond percolation. The probability measure is the product measure

$$P = \prod_{i \in I} \mu_i, \text{ where } \mu_i(\omega(i)) = \begin{cases} p, & \text{if } \omega(i) = 1, \\ 1 - p, & \text{if } \omega(i) = 0, \end{cases}$$

where $\omega(i)$ is the i:th component of $\omega \in \Omega = S^I$. Product measure has as *marginal measures* (*component measures*) $P_i = P \circ pr_i$ where $\circ$ denotes the composition of two mappings and pr_i means the i:th component projection, $i \in I$, the *Bernoulli measures* that describe the case of having two possible values, the other with probability p and the other with probability $1 - p$, $0 \leq p \leq 1$.

For explanatory purposes, we shall speak of bond percolation, but much of the following is also true for the site percolation. For $\omega \in \Omega$ we say that if $\omega(e) = 1$, where $e \in E = I$, then the edge e is *open* and otherwise it is *closed*. Percolation theory studies the properties of the subgraph that has only open edges; important objects are the *open paths*, that is, paths is

the graph that use only open edges, and the *open clusters*, that is, the connected components of the subgraph that has only open edges. Two principal entities of percolation theory are the *percolation probability* θ, that is, the probability of the existence of an infinitely long open path from the vertex defined to be the *origin* (or equivalently, the probability that the origin belongs to an *infinite cluster*), and the *mean cluster size* χ that is the expectation value of the number of vertexes in an open cluster in which the origin belongs. The third major entity is the *number of open cluster per vertex* κ, which is the expectation of the inverse of the number of vertexes in an open cluster containing the origin. Also the *connectivity functions* $\tau(x_1, \ldots, x_n)$, being defined as the probability that vertexes $x_1, \ldots, x_n$ are in the same connected component are of interest, giving rise to so-called *correlation length* ξ which is very important in the *scaling theory* (to be defined below). The distribution of the size and the border size of the open cluster containing origin have also interested physicist. Note now that if the lattice is *translation invariant* (meaning that the lattice looks the same as viewed from any vertex) and so is the product measure (meaning the the probabilities of the edges being open are similar in all parts of the lattice), the model as a whole is translation invariant, meaning that any probability of interest is not changed when we change the vertex under consideration; that is why the probabilities are usually defined as some special point, the origin, being the vertex mentioned in the definition of the probabilities, but the choice of the origin is arbitrary, since the values of the probabilities are the same for all vertexes in the case that the underlying graph is a translation invariant lattice, id est, in the case all the vertexes really are in similar position. Note that this is not always the case, for example if the underlying graph is only "lattice-like" in the sense that there is a finite number of "types" that the vertexes can be divided into by their position in the graph such that two vertexes of the same type are equivalent.

The main results of percolation theory are that there exist a critical value of the edge-probability p, called the *critical probability* p_c, such that when $p < p_c$ (called *subcritical phase* or *subcritical percolation*), there almost surely are no infinite open clusters and in fact the probability of existence of an open cluster which is of diameter n (meaning that the two most-distant vertexes of the cluster have graph distance n) diminishes exponentially when $n \to \infty$, and that when $p > p_c$ (called *supercritical phase* or *supercritical percolation*), there almost surely is an unique infinite open cluster. Let us note that it has been proven that in two dimensions, there cannot be an infinite open and closed cluster simultaneously in percolation, but in three dimensions there can [13]. The calculation of the value of the critical probability for different lattices and for different dimensions has also been important subject. Not much is known as to the numerical values of the critical probability; for example, in the cubic lattice bond percolation it is only known that $p_c(1) = 1$ (trivially) and $p_c(2) = \frac{1}{2}$ (famous theorem of 1980 by *Harry Kesten*, who extended the 1960 results of *Theodore Edward*

Harris) where we denote the critical probability $p_c(d)$ where d is the dimension, so not even the physically most interesting case $d = 3$ has been solved. Lately the interest of researchers in the subject has moved into proving the conjectures that the *critical percolation* (when $p = p_c$) in plane is *conformally invariant*, meaning that if percolation is performed on some subgraph of $\mathbb{L}^d$ such that the subgraph is formed by vertexes and edges inside a simply connected plane domain D, then, if this domain is mapped conformally to another domain D', and percolation is performed in the subgraph defined by this new domain, then the processes in the two domains are related by the conformal map (here we expect the domains D, D' to be large in the sense that they both contain many vertexes and edges). Technically this holds only when we study the "*scaling limit*" measure of planar curves that bound the clusters formed by percolation; the "scaling limit" is defined by a limit process where we study percolation and open paths in lattices with decreasing *mesh* or edge length; that is, the lattices connected with the vertex sets $\delta\mathbb{Z}^d$, where $\delta \to 0$.

In the case of critical percolation it is also conjectured that no infinite cluster exists and that the tail of the open cluster C at the origin diminishes as a power, that is, the probability that C contains at least n vertexes behaves such that

$$\lim_{n\to\infty} \frac{\log P(|C| \geq n)}{\log(n^{-\frac{1}{\delta}})} = 1,$$

where $\delta > 0$ is a so-called *critical exponent* (the symbol δ is specifically reserved as the name of this exponent in percolation theory); an open conjecture is that $\delta = \frac{91}{5}$ in two dimensional square lattice percolation. There are also other *critical exponents* that appear as we study the behaviour of $\theta(p)$ and $\chi(p)$ and the *mean number of clusters per site* function (C is again the open cluster at origin)

$$f(p) = \sum_{n=1}^{\infty} \frac{1}{n} P(|C| = n),$$

which is the analog of the free energy of statistical physics in percolation, and other objects introduced above as $p \to p_c$. Typically the above functions are analytic or at least smooth in $p \in [0, 1]$ except at the point $p = p_c$; the non-regularity of these functions at the critical point is directly related with the type of phase transition happening in percolation, and the study of the behaviour of these functions near criticality (p near p_c) comprises the *scaling theory*, which has been suggested to mathematicians by physicist; much of the theory remains without mathematically rigorous foundations. One of the predictions of scaling theory is that the critical exponents do not depend on the lattice, but only on the dimension, whereas the value of the critical probability does depend on the lattice in question; this phenomenon is called *universality* and it is a major open problem to prove this.

Let it be noted that although the bond and site percolation models look similar, it can be shown that each bond model can be reformulated as a site model by changing the lattice, but the converse is not true. There also exist many extensions of percolation like the so-called *mixed model*, in which both vertexes and edges can be declared open/closed. Another extension are the *inhomogeneous models* in which the probability of an edge or vertex being open does not need to be constant but it can depend on the edge or vertex in question. The probability of different edges or vertexes being open need not be independent of the other edges or vertexes, which gives rise to so-called *dependent models*. There is also a model, called *long-range model*, in which we allow "jumps" in the paths such that the path need not proceed from one vertex to its lattice neighbour but can jump to a vertex far away. The special case of *oriented percolation* (or *directed percolation*, *DP*) means percolation performed in an oriented graph (where each edge is oriented); the only difference to normal percolation is that in constructing paths on the graph we must take into consideration the orientation of the edges (meaning that we can only go from vertex a to b if $(a, b) \in E$; it is not enough that $(b, a) \in E$). There are also continuum-type percolation models whose basic objects of study are random geometric graphs. The most basic model is the *Gilbert disc model* or *Boolean model* introduced by *Edgar Nelson Gilbert* in 1961; in the model one "drops" discs of radius r at plane in random locations such that the center points are distributed as a *Poisson* (*point*) *process*. The random graph is made by taking the centres of the discs as vertexes and connecting those vertexes whose distance is less than r by edges. Then the connectivity properties of this random graph are studied. This model can be used to study for example a network of transceivers placed at random locations. Another known continuum percolation model is the Voronoi percolation model, which is defined using Voronoi tessellations (named after *Georgy Feodosevich Voronoy* who studied them in 1908), which are also known as *Dirichlet domains* or *Dirichlet tessellations* (as they were studied by *Johann Peter Gustav Lejeune Dirichlet* in 1850) or *Thiessen polygonalizations* (rediscovered by *Alfred H. Thiessen* in 1911); also certain other names for these tessellations are used in other fields of science [111]. The model is defined as follows: take a Poisson point process, and to each point assign a *Voronoi cell* (or *Voronoi polygon* or *Dirichlet polygon*) of those points of the plane that are closer to this point than any other Poisson point, thus forming a *Voronoi tessellation* (or *Poisson-Voronoi tessellation*) of the plane. Form a random graph by taking the Poisson points as vertexes, and connect with edges those points whose Voronoi cells have a common boundary edge. The *Voronoi percolation* model is defined as the site percolation on above defined random graph, which is equivalent to *face percolation* (or *cell percolation*) on the *Voronoi diagram* (also known as *Dirichlet diagram*), which is the diagram of the Voronoi tessellation; face percolation just means that each face of the tessellation is named *open* with some constant probability independently of other faces,

and *closed* otherwise, and faces sharing common border are taken to be neighbours when one studies the connectivity-related properties of the random configuration on the diagram. Voronoi percolation is an example of percolation on *random environment*. Let us mention that the Voronoi tessellations have many applications to physics and other sciences. They are used for example in crystallography, where the model is known as the *Meijering model* (or slightly modified it is known as the *Johnson-Mehl model*). The physics interest lies in characterizing the shape of the "typical" Voronoi cell. Beyond the models introduced above there are many other extensions of, and models related to, percolation, so the above presentation is by no means exhaustive, but it is instead meant as an introduction to some of the different ways the percolation model can be extended.

Percolation theory is the basis of the *disordered media-theories* in physics, being the most simple model of random media. It has been used also in the study of interacting random systems, in which many models cannot be solved as of yet, but information about these models can be acquired by studying the more simple model of percolation. For example, percolation is closely related to the field of *disordered* (*random*) *electrical networks*, in which an electrical network is formed through random process similar (sometimes actually completely equal) to percolation (for example, one can study the resistance of a block of material that has been constructed by mixing two substances that do not form a mixture but are present as chunks), and in the study of *ferromagnetism* as described below when we consider the Ising model and the random cluster model. Percolation has also been modified to model the spread of diseases in an orchard or other plantation; the plants occupy the vertexes of a lattice, as they often do in a plantation, and an infected plant infects its neighbouring plants with probability p dependent on a parameter d, the distance of plants, making the situation look a little like bond percolation (the event of an infection being the open bond). The distance of plants must be chosen so that p is below the critical probability for the disease not to contaminate macroscopic portions of the plantation. This model can be developed further by taking chronological aspects into consideration to model forest fires in plantations (this leads to a model aptly named *forest fire model*). Percolation has applications also to the production of electrical circuits. There have even been relations of percolation theory to infinite particle system-studies and reliability theory.

5.7. **Ising model.** The *Ising model*, also known as *Ising spin model*, *spin lattice*, *Ising lattice* and *Lenz-Ising model*, is a model that stems from the studies of ferromagnetism in physics. It was introduced by *Wilhelm Lenz* in 1920 [60], who gave it to his student *Ernst Ising* as a problem for his thesis in 1922; the thesis was completed in 1924 and published in a journal in 1925 [91], [84], [45], [46], [47]. It is the standard model for phase transition between ferromagnetism and paramagnetism, and it has been applied to

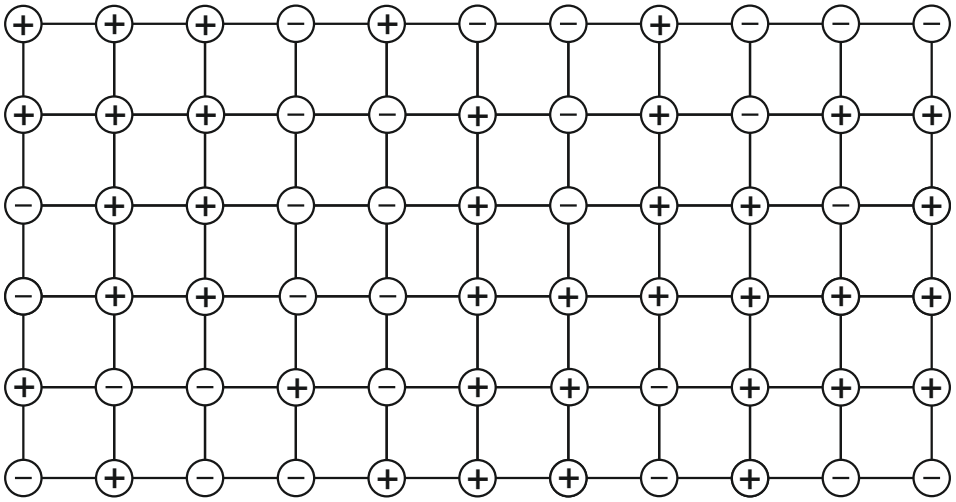
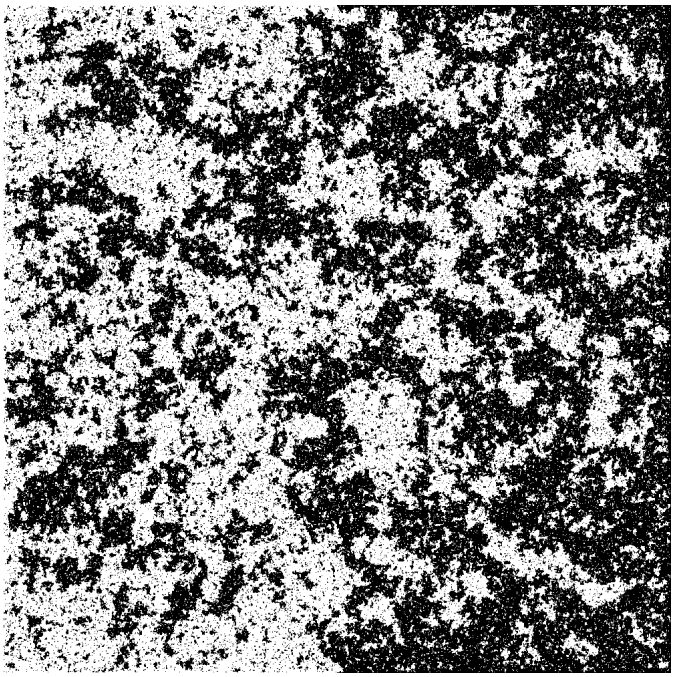


FIGURE 7. A configuration of the Ising model on a rectangular subgraph of the square lattice in two dimensions (a diagrammatic figure).

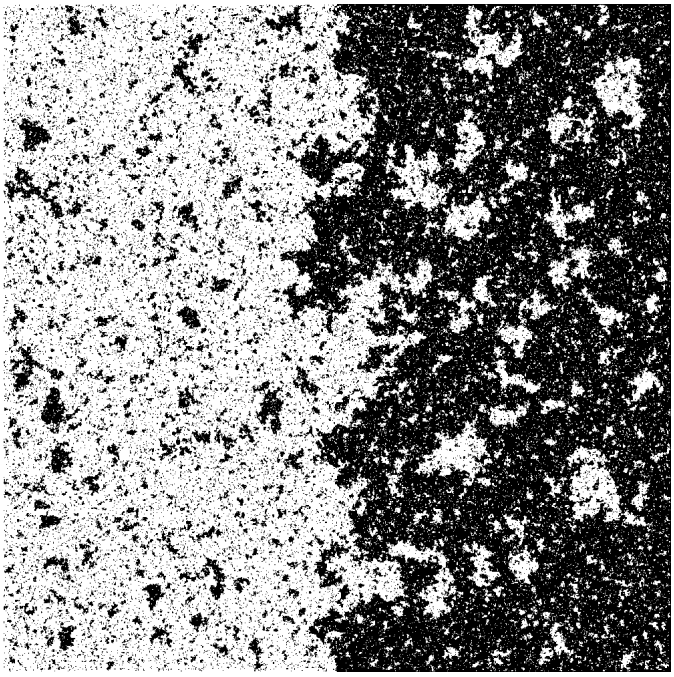
many other problems in many other fields; [65] declares that there has been over 15 000 articles published on the Ising model.

A material is a *ferromagnet* or *ferromagnetic* (the name stems from the fact that iron is this kind of material) if, when exposed to a magnetic field, it develops some additional magnetic force through arranging its molecules in a suitable way (the material *magnetizes*) and retains this magnetic organization even after the external magnetic field is switched off thus becoming a magnet; this is called *spontaneous magnetization* or more accurately *residual magnetization* of the material. A substance that develops additional magnetic force when exposed to an external magnetic field but does not retain this magnetism after the external field is switched off is called a *paramagnet* or *paramagnetic*. There is a limit temperature for the residual magnetization of a ferromagnetic material exposed temporarily to external magnetic field to occur; this temperature is called *Curie-temperature* (or *Curie-point* or *critical temperature*), as it was observed by *Pierre Curie* in his 1895 thesis, although a similar phenomenon was observed before in 1832 by *Claude Servais Mathias Pouillet* ([36], section 1.3, page 6, footnote 3). Also at the Curie temperature, the behaviour of the material changes from that of a ferromagnet to that of a paramagnet. This is called the phase transition between ferromagnetism and paramagnetism with respect to temperature.

The Ising model is defined as follows (we present the physical interpretation in the case of the study of ferromagnetism; the Ising model is however applied to the study of many other phenomena, and in these contexts the interpretation of the model differs). First of all, we have some crystalline



(A) A simulation of the Ising model in square lattice just above criticality with so-called *Dobrushin boundary conditions*, black boundary conditions on the right boundary and white on the left boundary (the boundary, that is, the set of vertexes at the geometric boundary of the square domain, is divided into right and left piece in the middle of the square and vertexes on the right and left pieces of the boundary are fixed black and white, respectively). Figure credit: *Vincent Beffara*.



(B) A simulation of the Ising model in the square lattice just below criticality with same boundary conditions as in figure 8a. Figure credit: *Vincent Beffara*.

lattice; since the atoms and molecules of real magnetic materials are positioned in space in a crystalline lattice, we can interpret the vertexes of the lattice to be these molecules, and the edges between the vertexes can be interpreted to encode the interaction-relations between the atoms, id est, only atoms "connected by an edge" interact with each other (the idea is that the interaction between nearest neighbouring atoms is so much stronger than the interaction between atoms farther away from each other that the latter can be ignored). Then to each vertex (atom) we join values $+1$ or -1 (sometimes called *spins* or *charges*) which we can interpret as the *spins* or *magnetic moments* of the atoms; the values correspond to spin up and spin down. Which value is given to a vertex is random and the probability depends on the values of the neighbouring vertexes.

Technically, let $G = (V, E)$ be a finite graph corresponding to a part of the desired crystalline lattice (depending on the material modelled), and let $S = \{-1, +1\}$ and $\Omega = S^V$ be the configuration space or the space of *spin-configurations*. The appropriate probability measure has three parameters that depend on the external conditions and the material studied; they are β, J, h where $\beta = \frac{1}{T} \in]0, \infty[$ (T is the temperature) and $J \in [0, \infty[$ is a constant describing the interaction between two neighbouring atoms in the lattice, and $h \in \mathbb{R}$ is the *external magnetic field*. The probability measure is

$$P_{\beta,J,h}(\omega) = \frac{1}{Z_{\beta,J,h}} e^{-\beta H(\omega)} \text{ for all } \omega \in \Omega,$$

where $H : \Omega \to \mathbb{R}$ is the Hamiltonian and $Z_{\beta,J,h}$ is the normalizing constant:

$$H(\omega) = -J \sum_{\langle x,y \rangle \in E} \omega(x)\omega(y) - h \sum_{x \in V} \omega(x),$$

$$Z_{\beta,J,h} = \sum_{\omega \in \Omega} e^{-\beta H(\omega)}.$$

The second term in the Hamiltonian is known as the *Zeeman term*. Let us assume $h = 0$; note that this assumption makes the model invariant under the change of charges $+1 \to -1$, $-1 \to +1$. Now we see that since $\beta > 0$, the configuration ω is more probable if the $H(\omega)$ is small. Therefore since $J > 0$, we see that the configuration is more probable if many neighbouring atoms have like spins, so that $\omega(x)\omega(y) = +1$ and the sum $H(\omega)$ is as small as possible. This property of preferring like spins is called *ferromagnetism* of the model since it causes the physical phenomenon of *ferromagnetism*, since the magnetic forces of the microscopical like spins combine to form a macroscopic magnetic force, id est, a magnet. If $J < 0$, so that the system prefers opposing spins, then we say that the system is *antiferromagnetic*. If $J = 0$, the system is called *noninteracting* or *free spin system*, a simple system of statistical physics describing the behaviour (approximately) of a crystalline paramagnetic substance at low temperatures. The free spin system is an *ideal system*, that is, a system without any internal interactions;

it is usually solved as an example in the first course of statistical physics, see for example [2]. The case $J \leq 0$ is also called *nonferromagnetic*.

In more general Ising models, the strength of the interaction J need not be the same for every edge (so-called *anisotropic model*), but individual strength J_e, $e \in E$ is defined for every edge. Another generalization is to make the magnetic field term h random (h can also depend on the site of lattice), usually following either a Bernoulli or a Gaussian distribution; this model is called *random field* (*Ising*) *model* (*RFIM*). Also the interaction need not be only of the *nearest neighbour-type* (in which only the vertexes joined by an edge can interact) but the interaction can be extended to cover vertexes that are of distance less than or equal to n of each other in the graph metric; in this case the sum in the Hamiltonian is extended over all the vertexes of distance n apart, and each pair of vertexes considered in the sum can have their own interaction strength; the model is called *n:th nearest neighbour model*. If we extend the sum over all the vertexes, the model is called *infinite range model* [91].

By making some changes to the Ising model, one can modify it to the lattice gas or binary alloy model. In a *binary alloy model*, we consider the vertex having value +1 in the configuration to be occupied by an atom of type A and vertex having value −1 to be occupied by atom of type B. By making some changes into the Hamiltonian (taking into account the different ways in which the different type atoms can interact), we get the binary alloy model from the Ising model. In the *lattice gas model*, we view the lattice vertexes having value +1 in the configuration as being occupied by a single molecule of the gas, and vertexes with value −1 as being not occupied. By varying the Ising Hamiltonian with an added summation constant and making some agreements as to how the constants of the different models relate, one can use the Ising Hamiltonian as the lattice gas Hamiltonian. Lattice gas model is used to model, for example, the hydrogen absorption of metal surfaces [75]. The basic object of study in lattice gas models is the phase transition between "*solid*" *state*, which has segregated regions of occupation and vacancy in terms of the "atoms", and "*gas*" *state*, in which there is no segregation.

If the vertex spins have values $\{-1, +1\}$ and the Hamiltonian is taken to be (here without the Zeeman term, which can also appear)

$$H(\omega) = -\sum_{\langle x,y\rangle \in E} J_{\langle x,y\rangle}\omega(x)\omega(y),$$

and the interaction constants $J_{\langle x,y\rangle}$ are taken not as constants but as independent identically distributed random variables, called *bond* or *link variables*, then the model is called (*classical*) *Edwards-Anderson model* (*EA-model*). The model is called *Gaussian* (*model*) if the interaction constants are distributed as Gaussian variables, and the model is called $\pm J$ (*model*) or *Ising spin glass* (note that the same term is sometimes used, rather erroneously,

to refer to the standard Ising model) if the interaction constants have value J with probability p and $-J$ with probability $1 - p$.

The Edwards-Anderson model is a so-called *spin glass model* because the interaction constants that dictate the interaction between the spins in the lattice vertexes are random. Let us mention that many different models can be made into spin glass models by randomizing the edge-interaction parameters; these models are sometimes labelled with the name of the original model followed by the attribute "*spin glass*". The reason for the name spin glass comes from physics; in a normal glass, the positions of the atoms are random and not crystalline; in a spin glass model, the (magnetic) interactions between atoms are random (although the location of the atoms are not). The most important open questions about the spin glasses concerns their phase transition between metastable and normal phases, in particular whether there even exists a critical temperature below which the model favours metastable behaviour and above which the model acts normally, id est, there is no particular order in the configuration space.

Let us describe the probability measure of the Edwards-Anderson model in more detail as an example of a spin glass model. First one calculates the normal zustandssumme (or first one considers some fixed or *quenched* values of $J_{\langle x,y\rangle}$):

$$Z = \sum_{\omega \in \{-1,+1\}^V} e^{-\beta H(\omega)}$$

which is a function of $J_{\langle x,y\rangle}$; then one takes the so-called *configurational average* (denoted by $\langle\rangle$) of this to obtain the final total mass:

$$\langle Z \rangle = \int_{\mathbb{R}_1} \dots \int_{R_{|E|}} Z \prod_{\langle x,y\rangle \in E} dJ_{\langle x,y\rangle}$$

where by $dJ_{\langle x,y\rangle}$ we mean the measure of the interaction constants; for example, if the interaction constants are Gaussian variables, then

$$dJ_{\langle x,y\rangle} = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right) dx$$

where dx refers to the normal Lebesgue measure. Hence the probability measure can be seen as being the product measure of the counting measures in the spin variables and the Gaussian measures in the interaction variables with the exponentiated (and scaled with $-\beta$) Hamiltonian as a weight function. This indicates how the model can be generalized to vector spins (in which the spin values are vectors, the Hamiltonian is similar, the product of vectors being the dot product, and the measure in the spin space is not the counting measure but for example the Lebesgue measure on the unit sphere); these generalizations are also called the *Edwards-Anderson model*. Here one also sees the two levels of randomness in the model; the spin-level randomness and the random variable-level randomness; models in which

there are two levels of randomness such that the probability (physically, energy) of a (spin-)configuration is not deterministic but still contains some random factors are called *disordered models*.

The above exposition of models that are Ising model derived or related is not by any means complete; there are tens (if not indeed hundreds) of models that are related to or derived from the Ising model. Most of these models have pretty similar "framework", the biggest difference being the type of "spins" considered (id est, what type of objects are assigned to the vertexes of the lattice) and the precise form of the Hamiltonian function (many model differ by small changes in the Hamiltonian).

Ising model is, as has been said many times, the most studied model in statistical physics. The first major result after Lenz introduced the model in 1920 was the solution of the one-dimensional case by Ising in his thesis. The one-dimensional case does not admit a phase transition in the system, but the correlations between spins decay always (at all temperatures) exponentially, so there is no ordered phase. After having proved this result Ising conjectured that the same was true for other dimensions also. This negative result also prompted *Werner Karl Heisenberg* to develop his own model (to be introduced below). The conjecture of Ising was however proven false by *Rudolf Peierls* in 1933, as he developed what is now known as *Peierls argument*, also known as *contour argument* (also analogous arguments are know by these names), that is, a way to count the number of relevant paths and contours in the lattice, to prove that the Ising model has a phase transition in dimension two. Peierls also showed that a finite system cannot have a phase transition (a singularity in the free energies) but at the thermodynamical limit of infinite system, the Ising model displays a phase transition. Let it be noted that the results of Peierls are among the greatest results of statistical physics; that is, the observation (and indeed mathematically proven fact) that a finite system cannot have a phase change, but the phase change occurs at the infinite system limit, and the observation that the behaviour of the lattice models depend remarkably on the dimension of the model (which is also mathematically proven); in statistical physics, unlike in many other fields of physics, lower-dimensional models cannot be used to give even an approximate description of the behaviour of the system (depending of course how far one is ready to go on the approximation; generally though lower-dimensional models in statistical physics can give one only qualitative insights of the behaviour of larger-dimension systems, and no quantitative results whatsoever). In 1941, *Hendrik Anthony Kramers* and *Gregory Hugh Wannier* developed the *Kramers-Wannier duality*, a method to couple Ising models in the square lattice that have different temperature parameter (one model having low and the other having high temperature); using this method they were able to determine the point of phase transition exactly. The analytical description of the two-dimensional Ising model was given by *Lars Onsager* in 1944 using a technique now known as *transfer matrixes*. In 1949 Onsager also presented a formula for the spontaneous

magnetization in two-dimensional Ising model without proof; the proof for the formula was given by *Chen-Ning Franklin Yang* in 1952. In 1952 Yang and *Tsung-Dao Lee* proved the *Yang-Lee theorem* describing the nature of the phase change in infinite system limit.

One more remarkable and unexpected observation that physicists and mathematicians have learned from the studies of the Ising model is that lattice models in low dimensions (less than four) do not follow the *mean field theory*, in which the interactions between different spins are forgotten and instead it is perceived that each spin interacts only with a medial external field, the "*mean field*". Now it is known that this is exactly true in infinite-dimensional limit case, and as an approximation the mean field works (well enough) in high dimensions (greater than four) but not in the physically interesting low dimensional regime.

In the above we have listed the "classical properties" of the Ising model; however the research on the model has continued in the decades since the 1950s, and new methods have been utilized to study the Ising model (it is fair to say that all the methods of statistical physics applicable to the Ising model have been applied to it). These techniques involve techniques related to *quantum field theory*, *conformal field theory* and *renormalization theory*; however these techniques are to some extend still ongoing research and we will not describe them more thoroughly here. The latest technique to emerge in study of the Ising model in two dimensions (and other lattice models) is the *Schramm-Loewner evolution*-random curves and *Stanislav Smirnov*'s (*para*)*fermionic observables*, which we will return to later in this thesis.

5.8. **Potts model.** The (*Domb-*)*Potts model* is named after *Renfrey Burnard Potts*, who defined it in his 1951 thesis [103]; again, an excerpt of the thesis was published the following year [66], and this has lead to the confusion that the thesis itself would have been published in 1952. The model was suggested to him by his instructor *Cyril Domb*. The model is sometimes called *Ashkin-Teller* or *AT-model* or *Ashkin-Teller-Potts model* after *Julius Ashkin* and *Edward Teller*, who studied a variant with 2-dimensional lattice and four components (or $q = 4$ below) in 1943 [3]. The special case $q = \infty$ is called the *Kac model* [98] after *Mark* (*Marek*) *Kac*.

The model is constructed like the Ising model, but unlike the Ising model which allows the vertexes to have one of only two possible values, the (*q-state*) *Potts model* (also called *scalar Potts model*) allows the value assigned to each vertex to be selected from any number $q \geq 2$ of possible values. So for Potts-model $S = \{1, \ldots, q\} \subseteq \mathbb{N}$ and $\Omega = S^V$. The probability measure is the thermodynamic measure with parameters β, q such that the Hamiltonian is

$$H(\omega) = -\sum_{\langle x,y\rangle \in E} \delta_{\omega(x),\omega(y)}, \tag{5.1}$$

where $\delta_{i,j}$ is the Kronecker delta. Again we see that the model favours configurations that have many neighbouring vertexes that have like values. When $q = 2$, and the possible values assigned to vertexes are defined to be $+1$ and -1, we see that the Potts model becomes the Ising model with parameters $\beta, J = \frac{1}{2}, h = 0$, since $\delta_{\omega(x),\omega(y)} = \frac{1}{2}(1 + \omega(x)\omega(y))$ (note that the 1 in the Kronecker delta's formula above does not cause complications due to the fact that the normalizing constant cancels it).

There exist also a related model that was also defined by Potts called the *clock model* (or *planar Potts model* or *vector Potts model*), in which the construction is otherwise similar, but the Hamiltonian is

$$H(\omega) = J \sum_{\langle x,y\rangle \in E} \cos\left(\theta_{\omega(x)} - \theta_{\omega(y)}\right) \text{ for all } \omega \in \Omega, \tag{5.2}$$

where J is the interaction constant, and the θ_n, $n = 1, \ldots, q$ are the angles $\theta_n = \frac{n}{q} 2\pi$, $n = 1, \ldots, q$. The idea is that the spin of the atom in the vertex can take q possible values that are uniformly distributed vectors in the unit circle [98]. This model is sometimes said to be an *Abelian model* because the symmetry group of this model is Abelian (similarly some models are called *non-Abelian*).

There is also the *chiral clock model* (also called *q-state asymmetric clock model* [24]) defined in 1981 by *Stellan Östlund* (*Ostlund*) and *David A. Huse* in which the Hamiltonian is

$$H(\omega) = J \sum_{\langle x,y\rangle \in E} \cos\left(\theta_{\omega(x)} - \theta_{\omega(y)+\phi}\right) \text{ for all } \omega \in \Omega,$$

where ϕ is a phase factor (a constant).

Potts models are used on surface physics. Potts model can be used to study for example the absorption of a noble gas to a surface, like krypton to graphite. Graphite forms a hexagonal two-dimensional lattice, to the faces of which krypton absorbs. The krypton can have three positions (states) as it is absorbed, giving rise to a three-valued Potts chiral clock model [75], in which an additive (chiral) term depending on the edge over which the term in the sum of (5.2) is calculated, is added into the cosine function's parameter in (5.2). The so-called *Potts lattice gas* is similar to the lattice gas model discussed in the Ising model subsection; the state space of the Potts lattice gas model is $\{0, 1, \ldots, q\}$ in which the state 0 means empty; the Hamiltonian is of course modified accordingly. The Potts lattice gas has a connection to a variant of the random cluster model (which we present below) called *asymmetric random cluster model*.

The above presentation is partly based on a private communication between the writer and *Jacques H. H. Perk*.

5.9. **O(n)-model.** The *n-vector* or *O(n)-model* (sometimes called "*O(n) spherical model*" or something of that sort) is yet another generalization of the Ising model. The model was introduced by *Harry Eugene Stanley* [55], [73], [89] in 1968. Let $n \in \mathbb{N}$ and let $S^{n-1} = \{x \in \mathbb{R}^n \,|\, |x| = 1\}$.

Now the set $S = S^{n-1}$ and $\Omega = S^V$. The probability measure is the thermodynamic measure (that is, the weight of a configuration is $e^{-\beta H(\omega)}$) integrated over the $|V|$ $n-1$-spheres such that the integration measure on the spheres is the $n-1$-dimensional Lebesgue measure normalized to 1 over the unit sphere. So, to put it in another words, the zustandssumme is

$$Z = \int_{S_1^{n-1}} \dots \int_{S_{|V|}^{n-1}} e^{-\beta H(\omega(s_1,\dots,s_{|V|}))} dm(s_1) \dots dm(s_{|V|})$$

where $\omega = \omega(s_1, \dots, s_{|V|}) = (s_1, \dots, s_{|V|})$, m is the $n-1$-dimensional normalized Lebesgue measure on the sphere, and the Hamiltonian is

$$H(\omega) = \sum_{\langle x,y\rangle \in E} \omega(x) \cdot \omega(y) \text{ for all } \omega \in \Omega$$

where we take the dot product of $\omega(x) \in S^{n-1}$ and $\omega(y) \in S^{n-1}$. When $n = 0$, this model is the SAW-model, when $n = 1$, we see that we have the Ising model with parameters $\beta, J = 1, h = 0$. When $n = 2$, the O(n)-model is called the *X/Y-model* (or *XY model* or *classical XY model* or *classical rotor* (*rotator*) *model*) and when $n = 3$, it is called the (*classical*) *Heisenberg model*. The case $n = 4$ is also important as an approximate (so-called *toy model*) model for the Higgs sector of the Standard Model in particle physics [93].

5.10. **Fortuin-Kasteleyn random cluster model.** The following *random cluster model*, also known by the names *Fortuin-Kasteleyn random cluster model*, *Fortuin-Kasteleyn percolation* or *FK percolation* or even *FK model* for short [17] was invented by *Cees* (also called *Kees*) *M. Fortuin* (real name is *Cornelius Marius Fortuin*, but Cees is usually used in scientific context) and *Pieter* (*Piet*) *Willem Kasteleyn* around 1970 [27], [28], [29], [30], [50], [36], [97], [6], and is not to be mixed with another *FK model* [11], which is the *Frenkel-Kontorova model*, that is a model of particle systems in classical mechanics and does not relate to statistical physics. The Frenkel-Kontorova model has many derivatives (like the *zigzag FK model*, the *secondary FK model*, et cetera) that also are deterministic and do not relate to statistical physics.

Random cluster model is a very general lattice model that contains as submodels the percolation, UST, Ising and Potts models, or to be more precise, it allows a so-called *graphical representation* of these models. In statistical physics, many models are of vertex-type (not to be confused with the so-called *vertex-models* that are actually of edge-type); that is, the vertexes of the lattice are assigned with some objects that interact according to the lattice in a way described by the Hamiltonian. To make a graphical representation of this kind of model means somehow transforming it to an edge-type model allowing the use of geometrical arguments or "*graphical methods*". Because the random cluster model allows this transformation, the random cluster model is sometimes called "*FK representation*" of some other model, especially when the parameters of the random cluster model

are chosen so that it can be made to correspond to some other model (sometimes a shorter name for this model is used; namely, only the prefix "*FK*" is added to the name of the model represented).

Again, in a more technical note, let $G = (V, E)$ be a finite graph, and let $S = \{0, 1\}$ and $\Omega = S^E$. For a configuration $\omega \in \Omega$, let $O(\omega) = \{e \in E \,|\, \omega(e) = 1\}$ be the set of open edges, and let $C(\omega)$ be the number of connected components of the subgraph $(V, O(\omega))$. The connected components are also called *open clusters*. The random cluster measure with parameters $p \in [0, 1]$ and $q \in]0, \infty[$ is the following probability measure:

$$P_{G,p,q}(\omega) = \frac{1}{Z_{G,p,q}} q^{C(\omega)} \prod_{e \in E} p^{\omega(e)} (1 - p)^{1-\omega(e)}, \tag{5.3}$$

where again the $Z = Z_{G,p,q} = \sum_{\omega \in \Omega} q^{C(\omega)} \prod_{e \in E} p^{\omega(e)} (1 - p)^{1-\omega(e)}$ is the normalizing constant. Note that the measure differs from the product measure of percolation by inclusion of the term $q^{C(\omega)}$, and that due to this difference, the edges in the random cluster model are not independent as they are in the percolation model.

One can also introduce boundary conditions to the model; *boundary conditions* ξ are a set of open edges that link the boundary vertexes (in general, the boundary vertexes are just a set of vertexes of G defined as the boundary vertexes; however when G is a subgraph of some other graph, the boundary vertexes are usually taken to be those vertexes that have a neighbour in the larger graph that is not in G, but other definitions, especially definitions based on the geometry of the situation, so that boundary vertexes of the graph correspond to the geometrical boundary of the representation of the graph, are also in use) of the graph G; the idea is that if G is a finite part of an infinite lattice, then the connections taking place outside of G can be encoded to the boundary conditions ξ. Two special kinds of boundary conditions are the *free boundary conditions*, in which no new edges are added to G, and the *wired boundary conditions*, in which all the boundary vertexes of G are connected by open edges. In the equation (5.3), the use of boundary conditions is taken into account by setting the measure of the configuration to be equal to that of equation (5.3) with the term $q^{C(\omega)}$ replaced by $q^{C(\omega \cup \xi)}$ (and the similar replacement is of course made in the normalizing constant Z), where the $\omega \cup \xi$-configuration means the configuration ω with added boundary edges ξ. So boundary conditions affect the measure of a configuration through changing the number of clusters. However, the adding of edges to the configuration takes place only in the probability measure and not in the configuration itself, meaning that when one is for example studying the open paths of the configuration ω, one does not consider the edges of ξ as possible open edges for the paths, even if one is using the boundary conditions ξ. So especially even if one has the wired boundary conditions, it might still be the case that two boundary vertexes are not connected by an open path in the configuration ω. The use of boundary conditions is based on the *spatial* or *Domain Markov property* of random cluster model,

which basically states that the part of lattice outside the finite graph G can be represented just by appropriate boundary conditions ξ (which was our motivation to introduce the boundary conditions in the first place).

One sees that our definition of random cluster model is only applicable to a finite graph G; therefore, to expand the definition to infinite lattices, we need to again use the technique of weak limits of probability measures by taking a weak limit of random cluster measures with free boundary conditions. Let us note that actually the uniqueness of the infinite-volume measure with respect to the subgraphs of the lattice used in the weak limit process or with respect to the boundary conditions of the limiting measures is not trivial, and actually much effort has been put to research of this phenomena [36].

In random cluster model as in percolation, the main object of scientific interest (in the subject itself, that is, discounting various implications of the random cluster model to other models) is the *percolation probability*, that is, the probability of the existence of an infinite cluster from the origin. The behaviour of the percolation probability as a function of the parameters and especially the phenomenon of phase transition are the most interesting aspects of random cluster model; by phase transition we refer to, as in the case of percolation, the existence of a certain critical value of $p = p_c(q)$ such that when p is below this value (*subcritical*), there almost surely is no infinite cluster, and when p is above this value (*supercritical*), there almost surely is an unique infinite cluster. Also similarly to percolation, the properties of the random cluster model in subcritical, critical and supercritical domains, especially the exponential decay of the radius of clusters, are also scientifically interesting. There are many open problems in the field; for more information, we suggest [36]. Let us note in passing that a long-standing conjecture in the field was that for 2-dimensional lattices the value of the critical point is the self-dual point $p_c(q) = \frac{\sqrt{q}}{1+\sqrt{q}}$; this was recently proved for $q \geq 1$ in the square, triangular and hexagonal lattice by *Vincent Beffara* and *Hugo Duminil-Copin* utilizing the box-crossing property of random cluster model on the square lattice [8].

Percolation is retrieved from random cluster model by letting $q = 1$, and UST measure is obtained by taking the weak limit of measures ([36], section 1.5) as $p \to 0$ and $q \to 0$ such that $\frac{q}{p} \to 0$, but the relation to Ising and Potts models is more complex (when $q = 2$, we get a representation of the Ising model and when $q = 2, 3, 4, \ldots$ we get representations of the Potts models ([36], section 1.1)), and we explain it below. UCS measure is obtained as the weak limit when $p = \frac{1}{2}$ and $q \to 0$ ([36], section 1.5), and furthermore, as the weak limit of $p = q \to 0$, we get the UF-measure ([36], section 1.5).

5.10.1. *Edwards-Sokal coupling.* We shall present next the *Edwards-Sokal coupling* (*Robert G. Edwards*, *Alan David Sokal*) between the random cluster and the Potts models (note that coupling trivially extends to coupling

between random cluster and Ising models). Our presentation is the same as in [57], for the original article, see [20].

The Edwards-Sokal coupling links together the random cluster and Potts model such that one can view the questions concerning one model as questions concerning the other. This coupling was recognized already by *Cees Fortuin* and *Pieter Kasteleyn* themselves, and nowadays (and in part when Fortuin and Kasteleyn defined the model) the basic motivation for studying random cluster model is the connection to other models; basically this connection allows one to express, for example, questions of *correlation* (how much effect the spins on different vertexes have on each other) in the Potts models (that are the questions most interesting to physicist) as geometrical questions in the random cluster model. It has even been said that anything worth doing with the Ising model should be done in the random cluster setting; whether this is true to the letter can be disputed, but it serves to show the significance of random cluster model in statistical physics. For more general discussion, see [35].

We shall use the technique of modifying the zustandssumme, which basically means that we write the zustandssumme in alternative forms such that one can always read the weight of one configuration as a term of the zustandssumme. This technique is widely used in physics; in elementary cases as we are about to show, this technique is related to the mathematical technique of coupling in an obvious way.

First we shall manipulate the zustandssumme of the q-valued Potts model (we denote configurations $\sigma \in \Sigma = \{1, \dots, q\}^V$) to a so-called *high temperature expansion*-form:

$$\begin{aligned} Z_{\text{Potts}} &= \sum_{\sigma\in\Sigma} e^{-\beta H(\sigma)} \\ &= \sum_{\sigma\in\Sigma} e^{\beta J \sum_{\langle x,y\rangle\in E} \delta_{\sigma(x),\sigma(y)}} \\ &= \sum_{\sigma\in\Sigma} \prod_{\langle x,y\rangle\in E} e^{\beta J \delta_{\sigma(x),\sigma(y)}} \\ &= \sum_{\sigma\in\Sigma} \prod_{\langle x,y\rangle\in E} \left(1 + \left(e^{\beta J} - 1\right)\delta_{\sigma(x),\sigma(y)}\right). \end{aligned}$$

Now, as is standard, we define a new parameter $v = e^{\beta J} - 1 > 0$ and we denote $E_\sigma = \{\langle x, y\rangle \in E \mid \sigma(x) = \sigma(y)\}$; hence the zusstandssumme becomes

$$Z_{\text{Potts}} = \sum_{\sigma\in\Sigma} \prod_{\langle x,y\rangle\in E_\sigma} (1 + v) = \sum_{\sigma\in\Sigma} \sum_{E'\subset E_\sigma} v^{|E'|}. \tag{5.4}$$

Then we gather together all the terms corresponding to the same subset E'' of E (note: not E_σ). The number of these terms is simply the number of configurations in which same values of spins are assigned at least to every endpoint of E''. Since there are q possible spins and we can assign each connected component of the graph (V, E'') the spin in the vertexes of that component independently, then if there are $C(E'')$ connected components

of the graph (V, E''), then the number of acceptable spin configurations σ is by elementary combinatorics $q^{C(E'')}$. Hence the zustandssumme is

$$Z_{\text{Potts}} = \sum_{E'' \subset E} q^{C(E'')} v^{|E''|}. \tag{5.5}$$

The logic of calling the above form of the zustandssumme the *high temperature expansion* (sometimes called *Fortuin-Kasteleyn representation* of the zustandssumme, see [56], page 29) is that for high temperature T, id est, for small $\beta = \frac{1}{T}$, the parameter $v = e^{\beta J} - 1$ is close to zero, hence above equation can be made a power series with small parameter, which can be approximated by the few first terms.

Now that we have come so far, the coupling should be predictable from the above form of the zustandssumme; namely if we take as configurations of the random cluster model $\omega \in \Omega = \{0, 1\}^E$ the sets of open edges, that is we define ω so that $O(\omega) = \{e \in E \,|\, \omega(e) = 1\} = E''$ and we define the parameter $p = \frac{v}{1+v}$, then the zustandssumme comes to form

$$Z_{\text{Potts}} = \sum_{\omega \in \Omega} q^{C(\omega)} \left(\frac{p}{1-p} \right)^{|O(\omega)|},$$

from which we see that (RCM stands for "Random Cluster Model")

$$(1-p)^{|E|} Z_{\text{Potts}} = \sum_{\omega \in \Omega} p^{|O(\omega)|} (1-p)^{|E| - |O(\omega)|} q^{C(\omega)} = Z_{\text{RCM}}.$$

This would be the physicist "proof" of the coupling; but for mathematics purposes, we shall develop this coupling a bit further.

Definition 5.1. A mathematical *coupling of two probability spaces* (stochastic models et cetera) $(\Omega_i, \mathcal{F}_i, P_i)$, $i = 1, 2$ is a probability measure $\mathbf{P}$ on the product measure space of $(\Omega_i, \mathcal{F}_i)$'s such that the marginal distributions are P_1 and P_2; that is, for $A_1 \in \mathcal{F}_1$ and $A_2 \in \mathcal{F}_2$ we should have

$$\mathbf{P}(A_1 \times \Omega_2) = P_1(A_1)$$
$$\mathbf{P}(\Omega_1 \times A_2) = P_2(A_2).$$

Now we shall return to (5.4) and write it as follows:

$$Z_{\text{Potts}} = \sum_{\sigma \in \Sigma, \omega \in \Omega} v^{|O(\omega)|} \chi_{O(\omega) \subset E_\sigma}(\sigma, \omega),$$

where the characteristic function

$$\chi_{O(\omega) \subset E_\sigma}(\sigma, \omega) = \begin{cases} 1 \text{ if } O(\omega) \subseteq E_\sigma, \\ 0 \text{ otherwise,} \end{cases}$$

where $O(\omega) = \{e \in E \,|\, \omega(e) = 1\}$, $E_\sigma = \{\langle x, y \rangle \in E \,|\, \sigma(x) = \sigma(y)\}$.

Now we define the probability measure $\mathbf{P}$ on $\Sigma \times \Omega$ as follows (note that by defining the probability measure on each configuration of $\Sigma \times \Omega$, and

since the original measures are defined for each configuration on Σ and Ω, respectively, there are no technical problems with σ-algebras and so on)

$$\mathbf{P}(\sigma, \omega) = \frac{1}{Z_{\text{Potts}}} v^{|O(\omega)|} \chi_{O(\omega) \subset E_\sigma}(\sigma, \omega).$$

This equation is the *Edwards-Sokal coupling*. We have de facto calculated the marginals above in (5.4) and (5.5) using the zustandssumme-formalism:

$$\begin{aligned}
\mathbf{P}(\{\sigma\} \times \Omega) &= \sum_{\omega \in \Omega} \mathbf{P}(\sigma, \omega) \\
&= \sum_{\omega \in \Omega} \frac{1}{Z_{\text{Potts}}} v^{|O(\omega)|} \chi_{O(\omega) \subset E_\sigma}(\sigma, \omega) \\
&= \frac{1}{Z_{\text{Potts}}} \sum_{E' \subset E_\sigma} v^{|E'|} = P_{\text{Potts}}(\sigma),
\end{aligned}$$

and on the other hand

$$\begin{aligned}
\mathbf{P}(\Sigma \times \{\omega\}) &= \sum_{\sigma \in \Sigma} \mathbf{P}(\sigma, \omega) \\
&= \sum_{\sigma \in \Sigma} \frac{1}{Z_{\text{Potts}}} v^{|O(\omega)|} \chi_{O(\omega) \subset E_\sigma}(\sigma, \omega) \\
&= \frac{1}{Z_{\text{Potts}}} q^{C(\omega)} v^{|O(\omega)|} \\
&= \frac{(1-p)^{|E|}}{(1-p)^{|E|} Z_{\text{Potts}}} q^{C(\omega)} \left(\frac{p}{1-p} \right)^{|O(\omega)|} \\
&= \frac{1}{Z_{\text{RCM}}} p^{|O(\omega)|} (1-p)^{|E| - |O(\omega)|} q^{C(\omega)} = P_{\text{RCM}}(\omega),
\end{aligned}$$

where we note that $\sum_{\sigma \in \Sigma} \chi_{O(\omega) \subset E_\sigma}(\sigma, \omega) = q^{C(\omega)}$, because we are again counting how many configurations σ there is such that at least the endpoints of the edges of $O(\omega)$ have the same spins in σ. The parameters p, v and β are again related as

$$\frac{p}{1-p} = v = e^{\beta J} - 1.$$

Let us now give a theorem describing the conditional distributions of the above Edwards-Sokal coupling:

Theorem 5.1. *Given ω the conditional distribution of σ is that obtained by assigning independently and uniformly the spin-value $\{1, \ldots, q\}$ to each connected component of the graph $(V, O(\omega))$. Given σ the conditional distribution of ω is that obtained by independently declaring each edge $e \in E_\sigma$ open ($\omega(e) = 1$) with probability p and closed otherwise, and declaring each edge $e \in E \setminus E_\sigma$ closed.*

Proof. Let $\omega \in \Omega$ and $S \subset \Sigma$. Let $S' = \{\sigma \in S \,|\, O(\omega) \subset E_\sigma\} \subset S$ and $\Sigma' = \{\sigma \in \Sigma \,|\, O(\omega) \subset E_\sigma\} \subset \Sigma$. We calculate

$$\begin{aligned}
\mathbf{P}(S \times \Omega \,|\, \Sigma \times \{\omega\}) &= \frac{\mathbf{P}((S \times \Omega) \cap (\Sigma \times \{\omega\}))}{\mathbf{P}(\Sigma \times \omega)} \\
&= \frac{\sum_{\sigma \in S} \mathbf{P}(\sigma, \omega)}{\sum_{\sigma \in \Sigma} \mathbf{P}(\sigma, \omega)} \\
&= \frac{\sum_{\sigma \in S'} \mathbf{P}(\sigma, \omega)}{\sum_{\sigma \in \Sigma'} \mathbf{P}(\sigma, \omega)} \\
&= \frac{|S'| \frac{1}{Z_{\text{Potts}}} v^{|O(\omega)|}}{|\Sigma'| \frac{1}{Z_{\text{Potts}}} v^{|O(\omega)|}} = \frac{|S'|}{|\Sigma'|}.
\end{aligned}$$

Thus $\mathbf{P}(* \times \Omega \,|\, \Sigma \times \{\omega\})$ is the uniform measure on Σ'. The configurations $\sigma \in \Sigma'$ are precisely those who have any of the q spins assigned to each connected component of ω; there is $q^{C(\omega)}$ of them. Hence we have shown the first claim.

For the second claim, let $\sigma \in \Sigma$ and $\omega \in \Omega$. Now if any of the edges $e \notin E_\sigma$ were open in ω, then we would have $O(\omega) \not\subseteq E_\sigma$ and thus

$$\begin{aligned}
\mathbf{P}(\Sigma \times \{\omega\} \,|\, \{\sigma\} \times \Omega) &= \frac{\mathbf{P}((\Sigma \times \{\omega\} \cap \{\sigma\} \times \Omega)}{\mathbf{P}(\{\sigma\} \times \Omega)} \\
&= \frac{\mathbf{P}(\sigma, \omega)}{\mathbf{P}(\{\sigma\} \times \Omega)} \\
&= \frac{\frac{1}{Z_{\text{Potts}}} v^{|O(\omega)|} \chi_{O(\omega) \subset E_\sigma}(\sigma, \omega)}{\mathbf{P}(\{\sigma\} \times \Omega)} \\
&= \frac{0}{\mathbf{P}(\{\sigma\} \times \Omega)} = 0.
\end{aligned}$$

Hence the set of configurations ω consistent with σ is $\Omega' = \{\omega \in \Omega \,|\, \omega(e) = 0 \text{ for all } e \in E \setminus E_\sigma\}$. We can calculate

$$\begin{aligned}
\mathbf{P}(\Sigma \times \{\omega\} \,|\, \{\sigma\} \times \Omega) &= \frac{\mathbf{P}(\Sigma \times \{\omega\} \cap \{\sigma\} \times \Omega)}{\mathbf{P}(\{\sigma\} \times \Omega)} \\
&= \frac{\mathbf{P}(\sigma, \omega)}{\sum_{\omega' \in \Omega'} \mathbf{P}(\sigma, \omega')} \\
&= \frac{\frac{1}{Z_{\text{Potts}}} v^{|O(\omega)|} \chi_{O(\omega) \subset E_\sigma}(\sigma, \omega)}{\sum_{\omega' \in \Omega'} \frac{1}{Z_{\text{Potts}}} v^{|O(\omega')|} \chi_{O(\omega') \subset E_\sigma}(\sigma, \omega')} \\
&= \frac{v^{|O(\omega)|}}{\sum_{\omega' \in \Omega'} v^{|O(\omega')|}} \\
&= \frac{\prod_{e \in E_\sigma} v^{\omega(e)}}{\sum_{\omega' \in \Omega'} \prod_{e \in E_\sigma} v^{\omega'(e)}} \\
&= \frac{\prod_{e \in E_\sigma} v^{\omega(e)}}{\prod_{e \in E_\sigma} (1 + v)} \\
&= \prod_{e \in E_\sigma} \frac{v^{\omega(e)}}{1 + v}.
\end{aligned}$$

The single edge probabilities are as follows; for $e' \in E_\sigma$:

$$\begin{aligned}
\mathbf{P}(\Sigma \times \{\omega(e') = 1\} \,|\, \{\sigma\} \times \Omega) &= \sum_{\omega \in \Omega', \omega(e') = 1} \prod_{e \in E_\sigma} \frac{v^{\omega(e)}}{1 + v} \\
&= \frac{v\,(1 + v)^{|E_\sigma \setminus \{e'\}|}}{(1 + v)^{|E_\sigma|}} \\
&= \frac{v}{1 + v} = p.
\end{aligned}$$

Independence follows from the fact that $\mathbf{P}(\Sigma \times \{\omega\} \,|\, \{\sigma\} \times \Omega)$ is a product over the edges; therefore a similar kind of formula can be written for events $\Sigma \times \{\omega(e'_j) = 1; j = 1, \ldots, n\}$, $n = 1, \ldots, |E_\sigma|$. □

This idea that a random cluster representation of a spin model is found by finding a suitable way to assign spins to the clusters of the random cluster model is actually quite a general feature.

As a final example of the usefulness of the Edwards-Sokal coupling, let us calculate a Potts model correlation function in terms of the random cluster model. The *two-point correlation function* of the Potts model is $C(x, y) = E_{\text{Potts}}(\delta_{\sigma(x),\sigma(y)}) - \frac{1}{q}$, where by E_{Potts} we denote the expectation with respect to the Potts measure. The term $\frac{1}{q}$ is subtracted because if the spins σ were chosen completely independently, we would have $E_{\text{Potts}}(\delta_{\sigma(x),\sigma(y)}) = \frac{1}{q}$ for $x \neq y$. We hope to express this correlation in the random cluster model as the probability of connection between x and y, id est, the property that x and y are connected by an open path (which is the most natural question to ask in random cluster model); we write this as $x \leftrightsquigarrow y$. The probability

$P_{\mathrm{RCM}}(x \leftrightsquigarrow y)$ is sometimes called *connectivity function*. Now we calculate the relation between the correlation function and the connectivity function.

Theorem 5.2.

$$C(x,y) = \left(1 - \frac{1}{q}\right) P_{RCM}(x \leftrightsquigarrow y).$$

Proof. Let us manipulate the two-point correlation function

$$\begin{aligned}
C(x,y) + \frac{1}{q} &= E_{\mathrm{Potts}}(\delta_{\sigma(x),\sigma(y)}) \\
&= P_{\mathrm{Potts}}(\sigma(x) = \sigma(y)) \\
&= \mathbf{P}(\{\sigma(x) = \sigma(y)\} \times \Omega) \\
&= \sum_{\omega \in \Omega} \mathbf{P}(\{\sigma(x) = \sigma(y)\} \times \{\omega\} \,|\, \Sigma \times \{\omega\}) \mathbf{P}(\Sigma \times \{\omega\}) \\
&= \sum_{\omega \in \{x \leftrightsquigarrow y\}} \mathbf{P}(\{\sigma(x) = \sigma(y)\} \times \{\omega\} \,|\, \Sigma \times \{\omega\}) \mathbf{P}(\Sigma \times \{\omega\}) \\
&\quad + \sum_{\omega \notin \in \{x \leftrightsquigarrow y\}} \mathbf{P}(\{\sigma(x) = \sigma(y)\} \times \{\omega\} \,|\, \Sigma \times \{\omega\}) \mathbf{P}(\Sigma \times \{\omega\}).
\end{aligned}$$

Now the first conditional probability equals one, since the vertexes in the same cluster of $(V, O(\omega))$ have the same spin (in conditional measure); the second conditional probability equals $\frac{1}{q}$ because one of the q spins is assigned independently to every cluster, and x and y are in different clusters of $(V, O(\omega))$. Hence we get

$$\begin{aligned}
C(x,y) + \frac{1}{q} &= \sum_{\omega \in \{x \leftrightsquigarrow y\}} 1 * \mathbf{P}(\Sigma \times \{\omega\}) + \sum_{\omega \notin \{x \leftrightsquigarrow y\}} \frac{1}{q} \mathbf{P}(\Sigma \times \{\omega\})) \\
&= P_{\mathrm{RCM}}(x \leftrightsquigarrow y) + \frac{1}{q} P_{\mathrm{RCM}}(\Omega \setminus \{x \leftrightsquigarrow y\}) \\
&= P_{\mathrm{RCM}}(x \leftrightsquigarrow y) + \frac{1}{q} \left(1 - P_{\mathrm{RCM}}(x \leftrightsquigarrow y)\right).
\end{aligned}$$

Hence we get

$$C(x,y) = \left(1 - \frac{1}{q}\right) P_{\mathrm{RCM}}(x \leftrightsquigarrow y).$$

So the claim is true. □

More generally, let us be interested in Potts model and of some *observable* $f : \Sigma \to \mathbb{R}$ (that is simply a function of the configuration of the Potts

model). The mean value satisfies

$$\begin{aligned} E_{\text{Potts}}(f) &= \sum_{\sigma\in\Sigma} f(\sigma)P_{\text{Potts}}(\sigma) \\ &= \sum_{\sigma\in\Sigma,\omega\in\Omega} f(\sigma)\mathbf{P}(\sigma,\omega) \\ &= \sum_{\omega\in\Omega}\left(\sum_{\sigma\in\Sigma} f(\sigma)\mathbf{P}(\sigma\,|\,\omega)\right)P_{\text{RCM}}(\omega) \\ &= \sum_{\omega\in\Omega} F(\omega)P_{\text{RCM}}(\omega) \\ &= E_{\text{RCM}}(F) \end{aligned}$$

where $F(\omega) = (\sum_{\sigma\in\Sigma} f(\sigma)\mathbf{P}(\sigma\,|\,\omega))$, $F : \Omega \to \mathbb{R}$. Note that we have $\mathbf{P}(\sigma\,|\,\omega) = q^{-C(\omega)}\chi_{O(\omega)\subset E_\sigma}$, so indeed we get another observable F in the random cluster model.

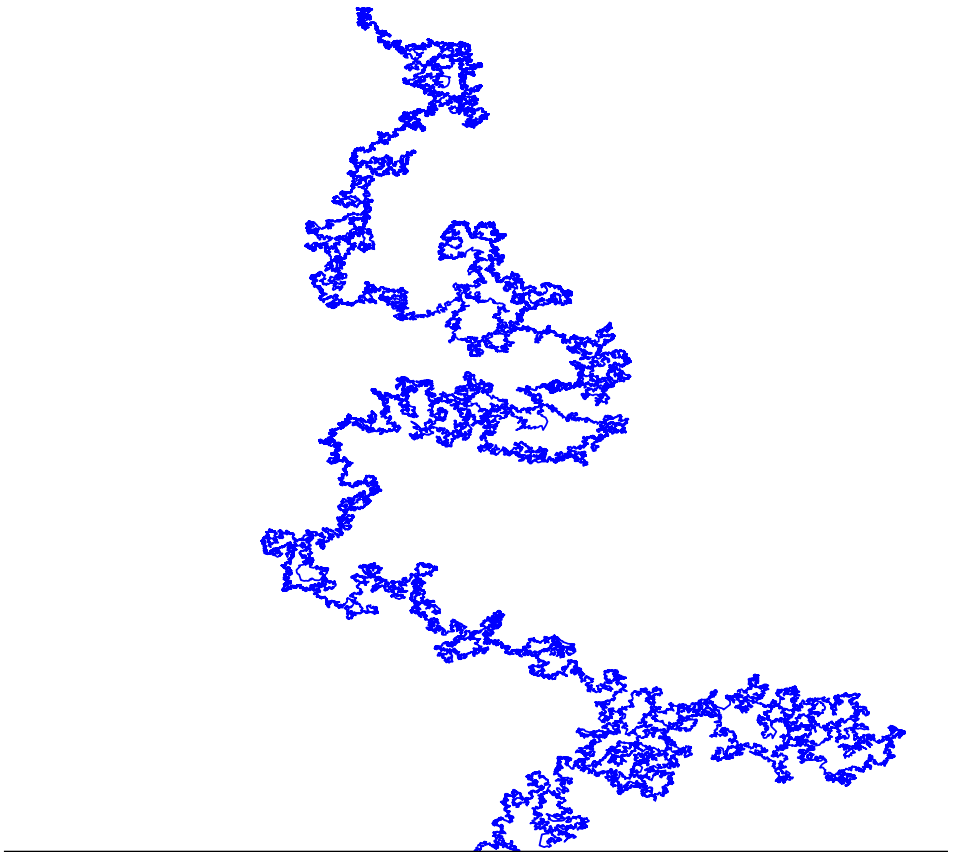


FIGURE 9. A figure of a SLE($\frac{16}{3}$)-simulation on the upper half plane starting from the origin. Figure credit: *Antti Kemppainen*.

6. SCHRAMM-LOEWNER EVOLUTION

The following exposition of Schramm-Loewner evolution is based on [51].

The *Schramm-Loewner evolution* (*SLE*), also known as *stochastic Loewner evolution*, is a random curve (or to be precise, a distribution of random curves) that arises as the "scaling limit" of many interfaces of many lattice models in two dimensions. That is, SLE is a way to define a probability measure in the space of (non-self-intersecting; the curves can touch themselves but not intersect) curves of the plane (taken to be the *complex plane* $\mathbb{C}$); we usually call the measure *SLE-measure*, and when we speak of *SLE-curves*, we mean the curves chosen according to this measure. The family of SLE-curves are parametrized by a real parameter $\kappa \geq 0$, and this is why we sometimes speak of the *SLE*(κ)*-curve*, by which we mean the SLE-distribution that is connected to parameter κ. For a *simply connected* domain $D \subsetneq \mathbb{C}$ (that is, a domain which has no holes in it) with two points

a,b chosen from the boundary of D, the SLE measure $P_{D,a,b}$ is defined in the space of curves $\gamma : [0,1] \to D$ from a to b that stay in the domain D.

The SLE-measure is desired to have the property of *conformal invariance*, that is, if $f : D \to D'$ where D' is a simply connected domain (not the whole plane) of complex plane, is a *conformal complex mapping* (that is, an injective analytic function), meaning that it preserves angles in the transformation of domain D into domain D', such that $f(a) = a'$ and $f(b) = b'$ (here we denote the extension of f to the boundary of D also with f), then the SLE-probability measure connected with the triple (D', a', b') should be $P_{D',a',b'} = f[P_{D,a,b}]$, where $f[]$ denotes the distribution, according to $P_{D,a,b}$, of $f \circ \gamma$, where γ is a random curve from a to b in domain D. Now we remember that the *Riemann mapping theorem* of complex analysis says that any two simply connected domains (excluding the whole plane) can be mapped to each other via conformal mappings, so that finding a conformal map f when two domains are given is not a problem. The conformal mapping f is not unique, but if in addition we demand that three points chosen from the boundary of one domain map to three points in the boundary of the other domain in fixed order, then the conformal mapping becomes uniquely defined.

The second property that the SLE-measure is desired to have is the so-called *domain Markov property*: if γ has distribution $P_{D,a,b}$, then $\gamma \restriction [t,1]$ where $0 \leq t < 1$ should have the distribution $P_{D\setminus\gamma([0,t[),\gamma(t),b}$. We shall define the SLE-measures below, and it will turn out that they do have the desired properties; furthermore, it can be shown that the different SLE(κ)-distributions are the only distributions in the space of (non-self-intersecting) curves that have the two above properties. Note also, that due to the conformal invariance, we see that the SLE-measure should be also invariant in scaling with a real constant $\lambda > 0$, meaning that the SLE should be statistically scale invariant; this would suggest that the "typical" curves according to SLE-measure should be *fractals*.

Because of the conformal invariance, it is enough to define the SLE-measure in the *upper half plane* $\mathbb{H} = \{z \in \mathbb{C} \mid \mathrm{Re}(z) > 0\}$; the SLE-measure in other domains is defined via the conformal mapping from $\mathbb{H}$ to the desired domain. Let γ be a curve that starts from origin, $\gamma(0) = 0$, and after that stays in the upper half plane, and ends up at infinity $\gamma(1) = \infty$. Let us introduce the unique (uniqueness and the series representation below are not proved here) conformal map $g_t : \mathbb{H} \setminus \gamma([0,t]) \to \mathbb{H}$ such that $g_t(z) = z + \frac{2t}{z} + \ldots$ at infinity. This family of conformal mappings satisfies the *Loewner differential equation* (that was introduced by *Charles Loewner* in 1923 when he was studying the so-called *Bieberbach conjecture*, a problem in complex analysis proposed by *Ludwig Georg Elias Moses Bieberbach* in 1916 and proved by *Louis de Branges de Bourcia* in 1984, and hence known

also as *de Branges's theorem* [110])

$$\partial_t g_t(z) = \frac{2}{g_t(z) - W(t)},$$

where $W(t) = g_t(\gamma(t))$ is the image of the tip of $\gamma(t)$ in the real axis. It is called the *driving function*. The Loewner differential equation is an ordinary differential equation for every $z \in \mathbb{H}$. Now the point is that the Loewner equation can be used as a mapping that maps a curve to the corresponding (continuous) driving function $W(t)$ one-to-one. Now we can also in some cases reverse the direction of mapping and map a driving function to the corresponding curve. If as the driving function we take a *stochastic process*, we get out of the equation a random curve. This is how SLE is defined: the SLE(κ)-curve from 0 to ∞ in $\mathbb{H}$ is the measure that is formed from the one-dimensional *Brownian motion* B_t, which has variance $E(B_t^2) = \kappa$, by using the Brownian motion as the driving function of the Loewner differential equation. So we have the distribution (measure) of the Brownian motion (in the space of continuous real functions) that through the Loewner equation gives us a measure in the space of continuous planar curves. It can be shown that the properties of conformal invariance and domain Markov property dictate the choice of the Brownian motion.

Many of the above described models, when critical, have certain curves that have the SLE-distribution as a weak limit when we take the *scaling limit* of the model, meaning that for example, in the cubic lattice, we start to decrease the distance between the vertexes from 1 to ever smaller values towards 0 (so we study the above introduced models in lattice $\delta\mathbb{L}^d$, which has as vertex-set $\delta\mathbb{Z}^d$ and as edge set $\delta\mathbb{E}^d = \{\langle x, y\rangle \,|\, |x - y| = \delta, x, y \in \delta\mathbb{Z}^d\}$, and take the limit $\delta \to 0$). The SLE is the scaling limit of some special curves in the above lattice models, most notably, the curves that form interfaces.

For example let us study the Ising model. Let us study the finite segment of the $\mathbb{L}^2$-lattice that is inside a rectangle R (by a *rectangle* we mean the planar region), denote it G, as the underlying graph, and we set following boundary conditions: let a be a corner vertex of G and b be the corner vertex of the opposite corner across; let $(a; b)$ be the curve in graph G (consisting of vertexes and edges) that forms the boundary (in the geometrical sense) of G from a to b in anti-clockwise direction; similarly let $(b; a)$ be the curve that forms the boundary of G from b to a in anti-clockwise direction. Now set all the vertexes in $(a; b)$ to have spin $+1$ and all the vertexes of $(b; a)$ to have spin -1; these are called *Dobrushin boundary conditions* (a two dimensional lattice subgraph that is the segment of the lattice contained in a simply connected planar domain and has these boundary conditions is called a *Dobrushin domain*). Now Ising configuration on G can qualitatively be divided into three areas: a $+1$-component consisting of the vertexes in $(a; b)$ and those vertexes to which one can define a path from $(a; b)$ such that all the vertexes encountered in the path have spin

+1 (we call this kind of a path a +1-*path*), a −1-component that consists of those vertexes with −1-spin and that can be linked to $(b;a)$ with a −1-*path*, and a third area consisting of separate "islands" of +1 and −1-spin clusters that cannot be linked to the boundary of G with +1 or −1-paths. Now the interface between the +1-component and −1-component is a simple random path in the lattice, and its distribution converges to SLE as the mesh of the lattice goes to zero. Similar kinds of interfaces can also be found in percolation by studying a part of the lattice and giving it a similar open/closed-boundary conditions. In different models these interface-curves have different SLE(κ)-curves to which they converge; for example, the above described Ising-interface-curve when the model is critical converges to SLE(3) in a number of different lattices, meaning that if we define a measure in the space of non-self-intersecting paths in the graph G_δ (understood as continuum-curves), which is the segment of $\delta\mathbb{L}^2$-lattice inside the rectangle R (as above), call it P_δ, then we have $P_\delta \Rightarrow$ SLE(3) as $\delta \to 0$. The above described convergence is proved by *Dmitry Sergeevich Chelkak* and *Stanislav Smirnov* [15]. In critical site percolation on the triangular lattice the interface converges to SLE(6) (as proved by *Stanislav Smirnov*). Loop-erased random walk converges to SLE(2) (shown by *Gregory Francis Lawler*, *Oded Schramm* and *Wendelin Werner*). The curve that encloses the uniform spanning tree in the dual lattice of $\mathbb{L}^d$ converges to SLE(8). Many other convergence results have been conjectured, and the matter is a subject of active research.

7. Box-crossing property

In two dimensional lattice models the *box-crossing property* is the property that with probability strictly greater than zero there exists a constant-value lattice path that either connects two sides of a rectangle of the lattice, a *rectangle* of a lattice being the subgraph of a lattice that is contained in a geometrical rectangle, or that circles the origin of the lattice in an *annulus*, that being the domain between two spheres (spheres are defined in the sense of graph-theoretical distance, so we are not talking about geometrical spheres), and that this probability is uniform in the size but not in the shape of the rectangle or annulus (so for example the width-length-ratio of the rectangle can affect this probability but not the size of the rectangle). This property can be formulated at least in some sense on those two-dimensional models where the concept of a "crossing of a rectangle" is applicable; actually even the non-discrete continuum percolation model has been proven to exhibit this property (let us also mention that in the case of the Voronoi percolation the box-crossing property has not been shown, only a weaker version is proven as of yet [10]). In the special case of models of (continuum) percolation and random cluster model in two dimensions, the box-crossing property's statement of the existence of the above mentioned paths with probability greater than 0 is sometimes called *Russo-Seymour-Welsh estimate*(*s*) (plural as there are two cases one is considering, the crossing of the rectangle and the contour in an annulus) or *RSW-estimate*(*s*) or *Russo-Seymour-Welsh theorem* or *RSW-theorem* (or something else with the names *Russo-Seymour-Welsh* or the letters *RSW* assigned to); the term *sponge crossing* or *sponge percolation* was also used in the 1980s literature. This is because the box-crossing property was originally considered only in the study of percolation-type models, and it was established in 1978 for the bond percolation in the square lattice by *Lucio Russo* [69] and independently by *Paul D. Seymour* and *Dominic James Anthony Welsh* [71]. When we say that box-crossing property has been established for some model, we mean that it has been mathematically proven that for some parameter-probability p_S and for parameter-probabilities greater than this $p > p_S$ the box-crossing property holds in this model. In many cases and certainly in the "normal" bond- and site-percolation on the "normal" lattices p_S is the critical percolation probability, so the critical and supercritical percolation models have the box-crossing property.

The reason the box-crossing property is scientifically interesting is that it can be used in study of the criticality of the lattice models, or at least those lattice models allowing a graphical representation. Indeed, most mathematical (as opposed to computer simulations and "proofs" of the critical probability value basing themselves on physical arguments) studies of the numerical value of the critical probability parameter of percolation and random cluster model on different lattices use the box-crossing property, and for example the proof of *Harry Kesten* [52] that the critical probability for

bond percolation in the square lattice and for the site percolation on the triangular lattice is $\frac{1}{2}$, which was generalized to triangular and hexagonal lattices by *John C. Wierman* [78] using star-triangle transformations, and the proofs of the criticality of the percolation on isoradial graphs by *Geoffrey Grimmett* and *Ioan Manolescu* [42], [40], and the proof of critical probability value ($p = \frac{\sqrt{q}}{1+\sqrt{q}}$) of the random cluster model for parameter value $q \geq 1$ (in which case the model has the so-called *strong positive association property*) on the square, triangular and hexagonal lattices by *Vincent Beffara* and *Hugo Duminil-Copin* [8], all rely on generalizing (from the 1978 square lattice bond percolation result) the box-crossing property to various lattices and models. Harry Kesten proved the box-crossing property for many homogeneous and inhomogeneous site-percolation models in the plane on lattices with sufficient symmetry (these results can also be applied to certain bond percolation models, as bond percolation can be considered as site percolation on suitable lattice; especially the results can be applied to the bond and site percolation on the square lattice) [53], and John Wierman extended Kesten's results for critical homogeneous percolation on a number of lattices (especially the triangular and hexagonal lattices). Geoffrey Grimmett and Ioan Manolescu proved the property for inhomogeneous (in some special cases homogeneous) percolation models on a very general class of isoradial graphs. In the case of the random cluster model, the box-crossing property was established for the square lattice by *Hugo Duminil-Copin*, *Clément Hongler* and *Pierre Nolin* [17] in the critical FK Ising case (parameter values $p = \frac{\sqrt{2}}{1+\sqrt{2}}$, $q = 2$), which was then generalized by Vincent Beffara and Hugo Duminil-Copin as described above. One can generalize these results of random cluster model to the Potts and Ising models through the Edwards-Sokal coupling. The box-crossing property is also frequently used on studies of other critical phenomena, like the critical exponents and their *scaling relations*, that is, relations between the critical exponents like in [54]. The box-crossing property is especially successful in the study of a class of critical exponents called *arm exponents*. The box-crossing property also plays a role in the study of conformal invariance of percolation; it is used in the proof of conformal invariance of critical site percolation on the triangular lattice by Stanislav Smirnov. Smirnov's proof, as well as other key applications (for example, proofs for the critical probability-values for edge percolation on the square and for site percolation on the triangular lattice) of the box-crossing property are presented in [38], sections 5.6, 5.7 and 5.8. All in all one could say the box-crossing property is one of the key ingredients of the study of percolation and random cluster model.

As the reader can see from the above cited publications, the research into this subject is very recent indeed, and therefore the "final word" on the matter cannot be given as of yet; indeed, the subject has evolved quite significantly even during the making of this thesis (the publishing of the above

cited articles [42], [40], [8] being one of the major changes that have happened during the making of this thesis). The proofs of the box-crossing property are based only on some geometrical continuity and symmetry-arguments, and therefore it is hypothesized that they can be extended to cover a wide number of different models and lattices. An open problem is whether the techniques of [40] can be used to extend the results of [8] to a class of lattices as wide as is considered in [40]. Another interesting question, though more speculative, is to consider what parts of the box-crossing technology, if any, can be used in three dimensions.

A note on terminology used in this paper: in 1978, Russo, Seymour and Welsh proved the box-crossing property for percolation in the square lattice by proving a theorem stating that one can obtain bounds to the probability of a horizontal crossing of rectangle in terms of the probability horizontal crossing of a rectangle half its width, from which the box-crossing property follows. As the square lattice case was generalized in the early 1980s, the box-crossing property in percolation models of all lattices became called the *Russo-Seymour-Welsh estimate*(*s*) (*theorem*(*s*)) or the *RSW-estimate*(*s*) (*-theorem*(*s*) (*property*(*ies*)). Also all the theorems, estimates and lemmas that were used to prove the box-crossing property or followed directly from the box-crossing property were called *Russo-Seymour-Welsh* or *RSW*-results in all lattices and models. In the modern literature the term *box-crossing property* has been introduced as above, and it is used by many authors (but not all; the terminology is not completely standard and some modern authors also call everything Russo-Seymour-Welsh or RSW). These authors reserve the term *Russo-Seymour-Welsh estimate* only to a theorem stating that in percolation in the square lattice (sometimes similar theorems on other lattices are referred to by the same name) one can obtain bounds to the probability of a horizontal crossing of rectangle in terms of the probability horizontal crossing of a rectangle half its width, and to the direct corollaries or antecedents of this theorem. The generalized estimates on other lattices than the square lattice are called *box-crossing estimates*. Also the similar results on other models than percolation are termed *box-crossing*-results by those how use the term box crossing; some writers (like for example [17]) call these type of results as *Russo-Seymour-Welsh*-type or *RSW*-type. As the reader can see, the terminology has not yet been stabilized as to whether to call things *box-crossing* or *Russo-Seymour-Welsh*, but it seems the modern approach of calling things *box-crossing* is taking over the previous terminology of *Russo-Seymour-Welsh*, and in this study we use the *box-crossing* terminology.

We will next prove the box-crossing property for homogeneous percolation in triangular and square lattices and for the random cluster model in the case $p = \frac{\sqrt{2}}{1+\sqrt{2}}$, $q = 2$ (the critical, self-dual FK Ising case). We will use a Russo-Seymour-Welsh type technique in the case of percolation, but although our technique is inherited from Russo, Seymour and Welsh,

the arguments we present are more modern and simple than theirs; the arguments are those by *Stanislav Smirnov*, who presented this argument for the triangular lattice in early 2000s and the application of the argument to the square lattice is due to the writer and *Antti Kemppainen* (closely resembling the more general treatise by Beffara and Duminil-Copin [8]). In the case of the random cluster model a completely different, more modern technique (making use of the fermionic observable introduced by Stanislav Smirnov in the 2000s) is used. Let us remark now that since the original, a little complicated method of Russo, Seymour and Welsh to prove the Russo-Seymour-Welsh theorem in the square lattice was published in 1978, there has been found two other method of proving the Russo-Seymour-Welsh theorem, namely that by Smirnov which we mentioned above (which is applicable at least to site percolation in the triangular lattice and the bond percolation on the square lattice) and also a simple method by *Béla Bollobás* and *Oliver Riordan* in the case of the square lattice. To the writers knowledge, the original arguments by Russo, Seymour and Welsh cannot be extended to other lattices than the square lattice; the generalizations of their results mentioned above use so-called *star-triangle transformations* (that are also called in electrical engineering with many names of the form *A-B transformation*, or sometimes *B-A transformation*, where $\mathrm{A} \in \{Y, T, wye, star\}$, $\mathrm{B} \in \{\Delta, \Pi, delta, pi, triangle, mesh\}$; in theoretical physics, the transformation is called *Yang-Baxter equation* and sometimes instead of the word transformation the words *transform*, *relation* or *equation* is used; the abbreviation is *STR*) to transport the box-crossing property from the square lattice to other lattices, so their argument itself has not been transported to other lattices. Also the writer has not seen Bollobás-Riordan argument being transported to other lattices than the square lattice (but of course one can transport the box-crossing property with star-triangle transformations as above).

7.1. **FKG-inequality.** We will need an auxiliary result in our proof of the box-crossing property called *FKG-inequality* (*Fortuin*, *Kasteleyn* and *Jean Ginibre*, 1971). In the case of percolation, or more generally in the case of independent and identically distributed random variables or most generally, in the case of product measure this inequality is called *Harris inequality*, and it is also known as *Kleitman inequality* or *lemma* in combinatorics and very rarely *Harris-Kleitman inequality*. The inequality is named after *Theodore Harris* who was the first to discover it in this limited setting in 1960 [86]. *Daniel J. Kleitman* proved the inequality in different setting independently six years after Harris [49]. However we actually need a stronger (actually it is an open question [16], whether the following inequality is equivalent to FKG-inequality) inequality for our studies of the random cluster model, the *Holley inequality* [86], also known as *Holley-Preston inequality* (especially when a slightly more general inequality is

concerned [67], [19]), that we can use to give us both FKG and Harris inequalities. In fact the Holley inequality itself is a special case of the so-called *Ahlswede-Daykin inequality* or *four functions inequality* (or rarely *AD-inequality* or *four-weight inequality*) [80] (which also has been generalized to measure spaces [16]). We shall prove this most general Ahlswede-Dayking inequality in a discrete case, relying on a proof from [16], and then we get the other inequalities as corollaries. We assume this approach because the proof of the most general inequality is, quite surprisingly, the easiest, and this approach ensures that we are able to utilize the above mentioned inequalities in sufficient generality as to satisfy the needs that arise from our studies of random cluster model (there are many proofs for the FKG- and Harris inequalities for percolation that are as simple as what we are about to show, see for example [34] or even the original article by Harris [44]). Let us also mention, although we do not utilize these inequalities, that there are also so-called *Griffiths inequalities* [88], also known as *Griffiths-Kelly-Sherman inequalities* or *GKS inequalities* (2 of them; these inequalities can be proved for many models, but originally they were formulated for the Ising model) that speak about ferromagnetic spin systems; these inequalities can be derived at least in the Ising model/Potts model case from the FKG inequality, see [37], [4], [31], as can their generalization (at least in some cases), the so-called *Ginibre inequality* [68]. Also the so-called *Chebyshev's sum inequality* and even a more general result by *Paul Seymour* and *Dominic Welsh* [32] is a consequence of the FKG inequality. Finally to convince the reader that we indeed are about to prove a very general inequality applicable in many scenarios, let us remark that in combinatorics and discrete mathematics, the Ahlswede-Daykin inequality can be used to give many other inequalities (besides the above mentioned) as corollaries, for example, the *Marica-Schönheim inequality*, *Fishburn-Shepp inequality*, *Daykin inequality* (*for distributivity*), the so-called *XYZ inequalities* and the *Seymour inequality* (which is actually a corollary of the Kleitman, id est, Harris inequality) and the many generalizations of the above [32], [25], [1].

Now let X be a finite set and let μ be a strictly positive probability measure on it (with the power set $\mathcal{P}(X)$ as the σ-algebra; note that the power set is partially ordered by inclusion). For any real-valued function f on $\mathcal{P}(X)$, let us denote

$$\mu(f) = \sum_{x \in \mathcal{P}(X)} f(x)\mu(x).$$

Definition 7.1. The probability measure $\mu : \mathcal{P}(X) \to \mathbb{R}$ on X is called *convex* (other terms in the literature include *log convex*, *supermodular*, *log supermodular* or *log-monotone*, or it said that the measure satisfies the *FKG lattice condition*) if for all $A, B \subset X$:

$$\mu(A \cup B)\mu(A \cap B) \geq \mu(A)\mu(B). \tag{7.1}$$

For two probability measures μ_1 and μ_2 we define that μ_1 *convexly dominates* μ_2 (it is also said in the literature that the probability measures fulfil the *Holley condition/criterion*), if for all $A, B \subset X$:

$$\mu_1(A \cup B)\mu_2(A \cap B) \geq \mu_1(A)\mu_2(B). \tag{7.2}$$

Definition 7.2. A function $f : \mathcal{P}(X) \to \mathbb{R}$ is called *increasing* (*decreasing*) if for all $A \subset B \subset X$

$$f(B) \geq f(A) \qquad (f(B) \leq f(A)).$$

Theorem 7.1. *(*FKG inequality*) If μ is convex and if f and g are increasing, then*

$$\mu(fg) \geq \mu(f)\mu(g). \tag{7.3}$$

(Sometimes it is said that f and g are covariant.*)*

Theorem 7.2. *(*Holley inequality*) Let μ_1 convexly dominate μ_2. If f is increasing, then we have*

$$\mu_1(f) \geq \mu_2(f),$$

and it is said that μ_1 stochastically dominates *μ_2.*

Theorem 7.3. *(*Ahlswede-Dayking inequality*) Let $f_i : \mathcal{P}(X) \to \mathbb{R}$, $i = 1, 2, 3, 4$ be non-negative functions satisfying for all $A, B \subset X$:*

$$f_1(A \cup B)f_2(A \cap B) \geq f_3(A)f_4(B). \tag{7.4}$$

Then we have

$$f_1(\mathcal{P}(X))f_2(\mathcal{P}(X)) \geq f_3(\mathcal{P}(X))f_4(\mathcal{P}(X)). \tag{7.5}$$

where $f_i(\mathcal{P}(X)) = \sum_{A \in \mathcal{P}(X)} f_i(A)$, $i = 1, 2, 3, 4$.

The above claim is enough for our purposes; let us just mention that the Ahlswede-Dayking inequality claims more; namely that for all $\mathbf{A}, \mathbf{B} \subset \mathcal{P}(X)$:

$$f_1(\mathbf{A} \vee \mathbf{B})f_2(\mathbf{A} \wedge \mathbf{B}) \geq f_3(\mathbf{A})f_4(\mathbf{B})$$

where for any $\mathbf{S}, \mathbf{T} \subset \mathcal{P}(X)$:

$$f_i(\mathbf{S}) = \sum_{x \in \mathbf{S}} f_i(x),$$
$$\mathbf{S} \vee \mathbf{T} = \{A \cup B \,|\, A \in \mathbf{S}, B \in \mathbf{T}\},$$
$$\mathbf{S} \wedge \mathbf{T} = \{A \cap B \,|\, A \in \mathbf{S}, B \in \mathbf{T}\}.$$

Now note that since a function is increasing if and only if its negative is decreasing, we can apply for example the FKG inequality to two decreasing functions yielding the same result; and if we apply the inequality to an increasing and a decreasing function, the direction of the inequality is reversed. Similar observations can be made on the Holley and Ahlswede-Daykin inequalities.

Now let us show that the Ahlswede-Daykin inequality does imply the Holley and FKG inequalities; note that by adding a constant (equal to the

least value) to the functions f and g of FKG inequality (or to function f of Holley inequality), both can be assumed positive (X is a finite set, id est, $\mathcal{P}(X)$ is finite), and if the inequalities hold for these functions with added constants, they hold for the original functions too (remember that the measures are probability measures). For the FKG inequality, one chooses $f_1 = fg\mu$, $f_2 = \mu$, $f_3 = f\mu$ and $f_4 = g\mu$. For the Holley inequality, one chooses $f_1 = f\mu_1$, $f_2 = \mu_2$, $f_3 = \mu_1$, $f_4 = f\mu_2$.

Also the Holley inequality implies the FKG inequality, as is seen when one takes $\mu_1 = \frac{g\mu}{\mu(g)}$ and $\mu_2 = \mu$.

Let us remark that convexity and convex domination are sufficient conditions for the claims of the FKG and Holley inequalities but not necessary. Also let us note that the above inequalities (FKG, Holley, Ahlswede-Daykin) hold not only in a powerset of a finite set, partially ordered by inclusion but also in a more general mathematical structures called finite *distributive lattices* (order-theoretic lattices, that are structures of *partial orders*) because every lattice of this sort is *lattice-isomorphic* to a *sublattice* of $\mathcal{P}(X)$ for some finite X, and the theorems can be easily transported to sublattices of $\mathcal{P}(X)$ by the introduction of characteristic functions of the sublattice; however, we do not delve deeper into lattice theory as it is enough for us to study just the power set of a finite set.

We need the following lemma:

Lemma 7.1. *Let s_1, s_2, s_3, s_4 and t_1, t_2, t_3, t_4 be non-negative real numbers such that $s_1 s_2 \geq t_1 t_2$ $s_3 s_4 \geq t_3 t_4$ and $s_2 s_3 \geq \max\{t_1 t_4, t_2 t_3\}$. Then*

$$(s_1 + s_3)(s_2 + s_4) \geq (t_1 + t_3)(t_2 + t_4).$$

Proof. Let us first assume $s_2 s_3 = 0$. Then

$$\begin{aligned}(s_1 + s_3)(s_2 + s_4) &= s_1 s_2 + s_1 s_4 + s_3 s_4 \\ &\geq t_1 t_2 + s_1 s_4 + t_3 t_4 \\ &= t_1 t_2 + t_1 t_4 + t_2 t_3 + t_3 t_4 + s_1 s_4 \\ &= (t_1 + t_3)(t_2 + t_4) + s_1 s_4 \\ &\geq (t_1 + t_3)(t_2 + t_4).\end{aligned}$$

where we note $0 = s_2 s_3 \geq \max\{t_1 t_4, t_2 t_3\} \geq 0$ and $s_1 s_4 \geq 0$.

Let us then assume $s_2 s_3 > 0$. Then

$$\begin{aligned}&(s_1 + s_3)(s_2 + s_4) - (t_1 + t_3)(t_2 + t_4) \\ &\geq \left(\frac{t_1 t_2}{s_2} + s_3\right)\left(s_2 + \frac{t_3 t_4}{s_3}\right) - (t_1 + t_3)(t_2 + t_4) \\ &= \frac{1}{s_2 s_3}\left((t_1 t_2 + s_3 s_2)(s_2 s_3 + t_3 t_4) - s_2 s_3 (t_1 + t_3)(t_2 + t_4)\right) \\ &= \frac{1}{s_2 s_3}(s_2 s_3 - t_1 t_4)(s_2 s_3 - t_2 t_3) \geq 0.\end{aligned}$$

□

Proof of Theorem 7.3 (Ahlswede-Daykin inequality). Let us denote the four functions as f_i^X, $i = 1, 2, 3, 4$. For arbitrary $Y \subset X$ define the function f_i^Y on $\mathcal{P}(Y)$, $i = 1, 2, 3, 4$ as follows:

$$f_i^Y(A) = \sum_{B \in \mathcal{P}(X \setminus Y)} f_i^X(A \cup B), A \subset Y.$$

We show that the f_i^Y satisfy (7.4) on $\mathcal{P}(Y)$. Clearly it suffices to show this when $Y = X \setminus \{z\}$ for some $z \in X$, because since X is finite, we can achieve any subset by just removing elements one by one and iterating the result for the case that just one element is removed. Note that the modified functions are such that this iteration is possible, id est, for arbitrary $Y \subset X$ the functions f_i^Y on $\mathcal{P}(Y)$ are the same as are those defined utilizing an arbitrary chain of sets $X = Y_0 \supset Y_1 = Y_0 \setminus \{y_1\} \supset Y_2 = Y_1 \setminus \{y_2\} \supset \ldots \supset Y_n = Y_{n-1} \setminus \{y_n\} = Y$ for some elements $y_j \in Y_{j-1} \subset X$, $j = 1, \ldots, n \in \mathbb{N}$, also $f_i^Y = f_i^{Y_n}$ is the same whether it is defined directly or having intermediary functions $f_i^{Y_j}$, $j = 0, 1, \ldots n$, $i = 1, 2, 3, 4$. This is because, if $i = 1, 2, 3, 4$, then as one can see, if f_i^Y is defined directly, then for any $A \subset Y$ in the sum defining $f_i^Y(A)$ there is a term $f_i^X(A \cup B)$ for all the possible sets B formed by choosing possible combinations of elements $y \in X \setminus Y$; whereas if f_i^Y is defined through the chain $Y_0, \ldots, Y_n = Y$, then for $A \subset Y$ one calculates in $f_i^Y(A) = f_i^{Y_n}(A)$ only two terms: whether to add the element y_n to A or not, and then one calculates $f_i^{Y_{n-1}}$ of sets thus formed (these sets are subsets of Y_{n-1}) and again one has got only two terms (per each value to be calculated): whether to include y_{n-1} or not. Continuing, one eventually comes to calculate $f_i^{Y_0} = f_i^X$ of sets formed by including or not including (independently) the elements $y_1, \ldots, y_n$ in A; but since $X \setminus Y = \{y_1, \ldots, y_n\}$, then one sees that this is the same as going through all the subsets B of $X \setminus Y$ and calculating $f_i^X(A \cup B)$; hence one can see that one does end up at the same functions on $\mathcal{P}(Y)$ (actually in the chain approach one just writes the sum of the direct approach a little differently).

So, let $Y = X \setminus \{z\}$ for some $z \in X$. Now the condition (7.4) follows directly from the above lemma 7.1 as one chooses (for $A, B \subset Y = X \setminus \{z\}$) $s_1 = f_1^X(A \cup B)$, $s_2 = f_2^X(A \cap B)$, $s_3 = f_1^X(A \cup B \cup \{z\})$, $s_4 = f_2^X((A \cap B) \cup \{z\})$, $t_1 = f_3^X(A)$, $t_2 = f_4^X(B)$, $t_3 = f_3^X(A \cup \{z\})$ and $t_4 = f_4^X(B \cup \{z\})$. Namely:

$$\begin{aligned}
&f_1^Y(A \cup B) f_2^Y(A \cap B) \\
&= \left(f_1^X(A \cup B) + f_1^X(A \cup B \cup \{z\})\right)\left(f_2^X(A \cap B) + f_2^X((A \cap B) \cup \{z\})\right) \\
&= (s_1 + s_3)(s_2 + s_4) \\
&\geq (t_1 + t_3)(t_2 + t_4) \\
&= \left(f_3^X(A) + f_3^X(A \cup \{z\})\right)\left(f_4^X(B) + f_4^X(B \cup \{z\})\right) \\
&= f_3^Y(A) f_4^Y(B).
\end{aligned}$$

Note that the lemma can be applied since

$$s_1 s_2 = f_1^X(A \cup B) f_2^X(A \cap B) \geq f_3^X(A) f_4^X(B) = t_1 t_2,$$

by the condition (7.4) applied to A,B,

$$s_3 s_4 = f_1^X(A \cup B \cup \{z\}) f_2^X((A \cap B) \cup \{z\}) \geq f_3^X(A \cup \{z\}) f_4^X(B \cup \{z\}),$$

by the condition (7.4) applied to $A \cup \{z\}$, $B \cup \{z\}$ and

$$\begin{aligned} s_2 s_3 &= f_2^X(A \cap B) f_1^X(A \cup B \cup \{z\}) \\ &\geq \max\{f_3^X(A) f_4^X(B \cup \{z\}), f_4^X(B) f_3^X(A \cup \{z\})\} \\ &= \max\{t_1 t_4, t_2 t_3\}, \end{aligned}$$

by the condition (7.4) applied to A, $B \cup \{z\}$ and $A \cup \{z\}$, B; note $X \ni z \notin Y \subset X \Rightarrow z \notin A$ and $z \notin B$.

Hence we have shown that the functions f_i^Y satisfy (7.4) on $\mathcal{P}(Y)$ for any $Y \subset X$, $i = 1, 2, 3, 4$. Now when we take $Y = \emptyset$, we have that

$$f_1^\emptyset(\emptyset) f_2^\emptyset(\emptyset) \geq f_3^\emptyset(\emptyset) f_4^\emptyset(\emptyset).$$

But now when we note $f_i^\emptyset(\emptyset) = f_i^X(\mathcal{P}(X))$, $i = 1, 2, 3, 4$, then we have the claim (7.5).

To prove the whole Ahlswede-Daykin inequality one would continue as follows: let $\mathbf{A}$, $\mathbf{B} \subset \mathcal{P}(X)$ and define $g_i^X : \mathcal{P}(X) \to [0, \infty[$ for $i = 1, 2, 3, 4$:

$$\begin{aligned} g_1^X(A) &= \chi_{A \in \mathbf{A} \vee \mathbf{B}}(A) f_1^X(A), \quad & g_3^X(A) &= \chi_{A \in \mathbf{A}}(A) f_3^X(A) \\ g_2^X(A) &= \chi_{A \in \mathbf{A} \wedge \mathbf{B}}(A) f_2^X(A), \quad & g_4^X(A) &= \chi_{A \in \mathbf{B}}(A) f_4^X(A) \end{aligned}$$

where $A \subset X$ and χ is the characteristic function. Then we have

$$\begin{aligned} g_1^X(P(X)) &= f_1^X(\mathbf{A} \vee \mathbf{B}), \quad & g_3^X(P(X)) &= f_3^X(\mathbf{A}) \\ g_2^X(P(X)) &= f_2^X(\mathbf{A} \wedge \mathbf{B}), \quad & g_4^X(P(X)) &= f_4^X(\mathbf{B}). \end{aligned}$$

So it is enough to show that the g_i^X, $i = 1, 2, 3, 4$ satisfy also the condition (7.4). Let $A, B \in \mathcal{P}(X)$; now since the functions g_i^X are non-negative, it is enough to study the case $A \in \mathbf{A}$, $B \in \mathbf{B}$. Then $A \cup B \in \mathbf{A} \vee \mathbf{B}$ and $A \cap B \in \mathbf{A} \wedge \mathbf{B}$, and then the condition follows from the fact that the functions f_i^X satisfy the condition (7.4), $i = 1, 2, 3, 4$. □

So now we have proven the FKG, Holley and Ahlswede-Daykin inequalities for probability measures and functions defined on the power set $\mathcal{P}(X)$ of some finite set X. Now to use these inequalities in the case of random cluster model or percolation, all that we need to do is to note that the configuration space $\Omega = \{0, 1\}^E$ can be understood as the power set of the set of edges $\mathcal{P}(E)$ by attaching to each $\omega \in \Omega$ the set $O(\omega) = \{e \in E \,|\, \omega(e) = 1\} \subset E$; so, when the underlying graph and hence set of edges E are finite, the measures and functions on Ω can be understood to be of the type we handled above. Also the *order* among different configurations, $\omega \leq \omega' \Leftrightarrow \omega(e) \leq \omega'(e)$ for all $e \in E$ for $\omega, \omega' \in \Omega$, can be seen as the inclusion order in $\mathcal{P}(E)$ (also, $O(\omega) \subset O(\omega') \subset E$). Let us also define for

two configurations $\omega_1, \omega_2 \in \Omega$ the configurations $\omega_1 \vee \omega_2$ and $\omega_1 \wedge \omega_2$ such that $O(\omega_1 \vee \omega_2) = O(\omega_1) \cup O(\omega_2)$ and $O(\omega_1 \wedge \omega_2) = O(\omega_1) \cap O(\omega_2)$.

Now we have to show that the percolation and random cluster measures satisfy the conditions of the Holley (7.2) and FKG (7.1) inequalities in a suitable way. So, let G be a non-empty finite graph, let $p \in [0, 1]$, $q \geq 1$ and choose any boundary conditions ϕ and ξ on G such that every connection present in ϕ is also present in ξ (we write $\phi \leq \xi$). We will show that $P^{\xi}_{G,p,q}$ convexly dominates $P^{\phi}_{G,p,q}$; this yields us the Holley inequality for random cluster measures with these boundary conditions. We note that by taking $\phi = \xi$, this amounts to saying that the random cluster measure with any boundary conditions satisfies the FKG lattice condition (see (7.1), (7.2)), yielding the FKG inequality. Furthermore by choosing $q = 1$, the random cluster percolation becomes normal bond percolation; this means that for a finite graph, the percolation measure also satisfies the FKG lattice condition (7.1); since the relevant σ-algebra is generated by finite cylinders (and since we are going to consider only cases where the underlying graph is finite, so in our cases the σ-algebra actually is a power set in the sense explained above), it is enough to consider the case where the set of edges E is finite in order to show that the percolation measure as a whole satisfies the (7.1), and therefore percolation satisfies the FKG or Harris inequality. Actually, this is a more general feature of percolation measure: in almost every case, it is enough to study only the case of finite underlying graph. The calculation we present is a variant of that presented in [33], [39].

So we need to show that $P^{\xi}_{G,p,q}$ convexly dominates $P^{\phi}_{G,p,q}$. Let $\omega_1, \omega_2 \in \Omega$; we claim that the following inequality holds:

$$P^{\xi}_{G,p,q}(\omega_1 \vee \omega_2) P^{\phi}_{G,p,q}(\omega_1 \wedge \omega_2) \geq P^{\xi}_{G,p,q}(\omega_1) P^{\phi}_{G,p,q}(\omega_2). \tag{7.6}$$

Let us denote the set of open edges in configuration ω as $O(\omega)$; we shall denote the set of edges of G by E. Note that the number of closed edges in configuration ω is $|E| - |O(\omega)|$. Let us manipulate the claim: the left-hand side of (7.6) can be written as:

$$
\begin{aligned}
&P^{\xi}_{G,p,q}(\omega_1 \vee \omega_2) P^{\phi}_{G,p,q}(\omega_1 \wedge \omega_2) \\
&= \left(\frac{1}{Z^{\xi}_{G,p,q}} q^{C((\omega_1 \vee \omega_2) \cup \xi)} \prod_{e \in E} p^{(\omega_1 \vee \omega_2)(e)} (1-p)^{1-(\omega_1 \vee \omega_2)(e)} \right) \\
&\quad \cdot \left(\frac{1}{Z^{\phi}_{G,p,q}} q^{C((\omega_1 \wedge \omega_2) \cup \phi)} \prod_{e \in E} p^{(\omega_1 \wedge \omega_2)(e)} (1-p)^{1-(\omega_1 \wedge \omega_2)(e)} \right) \\
&= \frac{1}{Z^{\xi}_{G,p,q} Z^{\phi}_{G,p,q}} q^{C((\omega_1 \vee \omega_2) \cup \xi) + C((\omega_1 \wedge \omega_2) \cup \phi)} p^{|O(\omega_1 \vee \omega_2)| + |O(\omega_1 \wedge \omega_2)|} \\
&\quad \cdot (1-p)^{|E| - |O(\omega_1 \vee \omega_2)| + |E| - |O(\omega_1 \wedge \omega_2)|} \\
&= \frac{1}{Z^{\xi}_{G,p,q} Z^{\phi}_{G,p,q}} q^{C((\omega_1 \vee \omega_2) \cup \xi) + C((\omega_1 \wedge \omega_2) \cup \phi)} p^{|O(\omega_1)| + |O(\omega_2)| - |O(\omega_1) \cap O(\omega_2)| + |O(\omega_1) \cap O(\omega_2)|} \\
&\quad \cdot (1-p)^{2|E| - (|O(\omega_1)| + |O(\omega_2)| - |O(\omega_1) \cap O(\omega_2)|) - |O(\omega_1) \cap O(\omega_2)|} \\
&= \frac{1}{Z^{\xi}_{G,p,q} Z^{\phi}_{G,p,q}} q^{C((\omega_1 \vee \omega_2) \cup \xi) + C((\omega_1 \wedge \omega_2) \cup \phi)} p^{|O(\omega_1)| + |O(\omega_2)|} (1-p)^{2|E| - |O(\omega_1)| - |O(\omega_2)|}.
\end{aligned}
$$

The right-hand side of (7.6) can be written as

$$
\begin{aligned}
&P^{\xi}_{G,p,q}(\omega_1) P^{\phi}_{G,p,q}(\omega_2) \\
&= \left(\frac{1}{Z^{\xi}_{G,p,q}} q^{C(\omega_1 \cup \xi)} \prod_{e \in E} p^{\omega_1(e)} (1-p)^{1-\omega_1(e)} \right) \\
&\quad \cdot \left(\frac{1}{Z^{\phi}_{G,p,q}} q^{C(\omega_2 \cup \phi)} \prod_{e \in E} p^{\omega_2(e)} (1-p)^{1-\omega_2(e)} \right) \\
&= \frac{1}{Z^{\xi}_{G,p,q} Z^{\phi}_{G,p,q}} q^{C(\omega_1 \cup \xi) + C(\omega_2 \cup \phi)} p^{|O(\omega_1)| + |O(\omega_2)|} (1-p)^{|E| - |O(\omega_1)| + |E| - |O(\omega_2)|} \\
&= \frac{1}{Z^{\xi}_{G,p,q} Z^{\phi}_{G,p,q}} q^{C(\omega_1 \cup \xi) + C(\omega_2 \cup \phi)} p^{|O(\omega_1)| + |O(\omega_2)|} (1-p)^{2|E| - |O(\omega_1)| - |O(\omega_2)|}
\end{aligned}
$$

where we note that $O(\omega_1 \wedge \omega_2) = O(\omega_1) \cap O(\omega_2)$ and $O(\omega_1 \vee \omega_2) = O(\omega_1) \cup O(\omega_2)$. Let now $p \in]0,1[$; for if $p = 0$, the only configuration with non-zero weight is the "empty" configuration; similarly, if $p = 1$, then the only configuration with non-zero weight is the "full" configuration; in these cases, both measures $P^{\phi}_{G,p,q}$ and $P^{\xi}_{G,p,q}$ give these configurations the weight one and other configurations the weight zero, id est, they are the same measure and hence the claim of convex dominance is true even when

$p = 0$ or $p = 1$. From the above we see that the claim is equivalent to

$$q^{C((\omega_1 \vee \omega_2) \cup \xi) + C((\omega_1 \wedge \omega_2) \cup \phi)} \geq q^{C(\omega_1 \cup \xi) + C(\omega_2 \cup \phi)}$$

which in the case $q \geq 1$ is equivalent to

$$C((\omega_1 \vee \omega_2) \cup \xi) + C((\omega_1 \wedge \omega_2) \cup \phi) \geq C(\omega_1 \cup \xi) + C(\omega_2 \cup \phi). \tag{7.7}$$

Now we prove inequality (7.7) by induction on $|O(\omega_1) \cup O(\omega_2)|$. If $|O(\omega_1) \cup O(\omega_2)| = 0$, inequality (7.7) is true as an equality (all of the configurations $\omega_1 \wedge \omega_2$, $\omega_1 \vee \omega_2$, ω_1, ω_2 are the same, namely the "empty" configuration). Suppose that (7.7) is true when $|O(\omega_1) \cup O(\omega_2)| \leq n$. Let ω_1 and ω_2 satisfy $|O(\omega_1) \cup O(\omega_2)| = n + 1$; we may assume $\omega_1 \neq \omega_2$, or else (7.7) is trivial. Hence there is an edge e that is in one of the sets $O(\omega_1)$, $O(\omega_2)$ but not in the other; let us assume first that $e \in O(\omega_1) \setminus O(\omega_2)$ (note that the other case is not similar due to different boundary conditions). Let us define ω_e^1:

$$\omega_e^1(e') = \begin{cases} \omega_1(e') \text{ for } e' \neq e, \\ 0 \text{ for } e' = e, \end{cases} \qquad \text{where } e' \in E.$$

Now by the induction hypothesis ($|O(\omega_e^1) \cup O(\omega_2)| = n$):

$$C((\omega_e^1 \vee \omega_2) \cup \xi) + C((\omega_e^1 \wedge \omega_2) \cup \phi) \geq C(\omega_e^1 \cup \xi) + C(\omega_2 \cup \phi). \tag{7.8}$$

Let us now define a characteristic function χ_e^ξ as the characteristic function of the set $\{\omega \in \Omega \mid \text{Endpoints of } e \text{ are in the same component in } \omega \cup \xi\}$. Now we have

$$\chi_e^\xi(\omega_e^1 \vee \omega_2) \geq \chi_e^\xi(\omega_e^1) \tag{7.9}$$

since $\omega_e^1 \leq \omega_e^1 \vee \omega_2$. Now adding the equations (7.8) and (7.9) we get

$$\begin{aligned} &C((\omega_e^1 \vee \omega_2) \cup \xi) + C((\omega_e^1 \wedge \omega_2) \cup \phi) + \chi_e^\xi(\omega_e^1 \vee \omega_2) \\ &\geq C(\omega_e^1 \cup \xi) + C(\omega_2 \cup \phi) + \chi_e^\xi(\omega_e^1). \end{aligned} \tag{7.10}$$

Noting that (we apply this to ω_1 and $\omega_1 \vee \omega_2$)

$$C(\nu_e \cup \xi) + \chi_e^\xi(\nu_e) = C(\nu \cup \xi) + 1 \text{ for } \nu \in \Omega \text{ such that } \nu(e) = 1$$

and that $\omega_e^1 \wedge \omega_2 = \omega_1 \wedge \omega_2$ and $\omega_e^1 \vee \omega_2 = (\omega_1 \vee \omega_2)_e$, equation (7.10) gives us the claim.

Then, let $e \in O(\omega_2) \setminus O(\omega_1)$. Let us define ω_e^2 as above:

$$\omega_e^2(e') = \begin{cases} \omega_2(e') \text{ for } e' \neq e, \\ 0 \text{ for } e' = e, \end{cases} \qquad \text{where } e' \in E.$$

Now by the induction hypothesis ($|O(\omega_1) \cup O(\omega_e^2)| = n$):

$$C((\omega_1 \vee \omega_e^2) \cup \xi) + C((\omega_1 \wedge \omega_e^2) \cup \phi) \geq C(\omega_1 \cup \xi) + C(\omega_e^2 \cup \phi). \tag{7.11}$$

Let us now define characteristic functions χ_e^ϕ, χ_e^ξ as above. Now we have as above

$$\chi_e^\xi(\omega_1 \vee \omega_e^2) \geq \chi_e^\xi(\omega_e^2) \tag{7.12}$$

since $\omega_e^2 \leq \omega_1 \vee \omega_e^2$. Now adding the equations (7.11) and (7.12) we get

$$C((\omega_1 \vee \omega_e^2) \cup \xi) + C((\omega_1 \wedge \omega_e^2) \cup \phi) + \chi_e^\xi(\omega_1 \vee \omega_e^2) \geq C(\omega_1 \cup \xi) + C(\omega_e^2 \cup \phi) + \chi_e^\xi(\omega_e^2).$$

Now we note that since $\phi \leq \xi$, $\chi_e^\phi \leq \chi_e^\xi$ (this is the only place we use this assumption) so that we have

$$\begin{aligned} (7.13) \qquad & C((\omega_1 \vee \omega_e^2) \cup \xi) + C((\omega_1 \wedge \omega_e^2) \cup \phi) + \chi_e^\xi(\omega_1 \vee \omega_e^2) \\ & \geq C(\omega_1 \cup \xi) + C(\omega_e^2 \cup \phi) + \chi_e^\phi(\omega_e^2). \end{aligned}$$

Noting that (we apply this to $\omega_1 \vee \omega_e^2$ with boundary conditions $\zeta = \xi$ and to ω_2 with boundary conditions $\zeta = \phi$)

$$C(\nu_e \cup \zeta) + \chi_e^\zeta(\nu_e) = C(\nu \cup \zeta) + 1 \text{ for } \nu \in \Omega \text{ such that } \nu(e) = 1$$

and that $\omega_1 \wedge \omega_e^2 = \omega_1 \wedge \omega_2$ and $\omega_1 \vee \omega_e^2 = (\omega_1 \vee \omega_2)_e$, equation (7.13) gives us the claim.

Hence the claim is true.

Now, by the calculations presented above, we have shown that for any non-empty graph G, any parameters $p \in [0,1]$, $q \geq 1$ and any boundary conditions ϕ and ξ, $\phi \leq \xi$, $P^\xi_{G,p,q}$ convexly dominates $P^\phi_{G,p,q}$. Hence since the random cluster measure is strictly positive (for $p \in]0,1[$; for $p = 0$ and $p = 1$ the measures $P^\phi_{G,p,q}$ and $P^\xi_{G,p,q}$ are the same measure as noted above and hence the claim of stochastic dominance is true) and satisfies (7.2), so by Holley inequality $P^\xi_{G,p,q}$ stochastically dominates $P^\phi_{G,p,q}$. It is said that the boundary conditions of the random cluster model satisfy the *monotonicity relation*.

The fact that the random cluster measures satisfy the Holley criterion (for $p \in [0,1]$, $q \geq 1$) is sometimes expressed by saying that the random cluster measure has the *strong positive association property*; the FKG inequality and the monotonicity of boundary conditions are said to be the consequences of this property.

Now we define that an event is *increasing* (*decreasing*) if and only if its characteristic function is increasing (decreasing); this is equivalent to saying that the non-empty event is increasing if and only if for all $\omega \in A$, when ω' is a configuration in which all the edges that are open in ω are also open (and there can of course be additional open edges in ω'), then $\omega' \in A$. Now the FKG inequality yields the following result that is also called *FKG inequality* (or *Harris inequality* in case of percolation):

Theorem 7.4. *(*Harris inequality*,* FKG inequality*) Let us consider percolation or random cluster models. For increasing events A and B it holds that*

$$P(A \cap B) \geq P(A)P(B)$$

for percolation measure and for random cluster measure with $q \geq 1$ with any boundary conditions.

The next observation is sometimes called *Holley inequality*:

Theorem 7.5. *(*Holley inequality*) For random cluster measures on non-empty graph G and parameters p, q ≥ 1 and for any boundary conditions φ, ξ such that φ ≤ ξ, if A is an increasing event,*

$$P^{\phi}_{G,p,q}(A) \leq P^{\xi}_{G,p,q}(A).$$

7.2. **Planar duality.** Let us now introduce the *planar duality* in the terms we shall apply it in our studies of random cluster model and percolation. We will talk about random cluster model here, because percolation is just random cluster model with $q = 1$. Furthermore we shall assume that there are no boundary conditions, id est, that we have free boundary conditions because the boundary conditions of the random cluster model make defining the planar dual (in an appropriate way) much more harder; we will return to this later. The construction we are about to present is standard in the literature; we have learned the argument from [57].

So let G be a finite planar graph. Let $G_d = (V_d, E_d)$ be the graph-theoretic dual graph of G; we shall define the dual model on this graph. We remind the reader that the vertexes of the dual graph are the faces of the original graph (including the infinite face) and the edges of the dual graph cross the edges of the original graph; for an edge $e \in E$ of the original graph, let us denote the unique edge of the dual graph crossing it by e_d.

Now let $\omega \in \Omega = \{0,1\}^E$ be a random cluster configuration, chosen according to random cluster measure in G with parameters p and q. The dual configuration space is $\Omega_d = \{0,1\}^{E_d}$, and the dual configuration to ω is defined as $\omega_d(e_d) = 1 - \omega(e)$ (open edges are intersected by closed dual edges and closed edges are intersected by open dual edges). This relation between original and dual configurations is a bijection. We note now that the number of open edges satisfy $|O(\omega)| + |O(\omega_d)| = |E| = |E_d|$. Next we shall calculate the probability of observing the dual configuration ω_d; it will turn out that the dual configuration is also distributed like a random cluster model configuration. Remember now the Euler's formula for general planar graphs: $|V| - |E| + F - C = 1$. We note that a face in the original configuration corresponds to a cluster in the dual configuration, id est, $F(\omega) = C(\omega_d)$. This is since $F(G) = |V(G_d)| = C((V(G_d), \emptyset))$, and when we study a configuration ω, we can think that we delete the closed edges of ω from G one-by-one, and every time we make this deletion, we either decrease the amount of faces of the original graph by one and we also decrease the number of clusters of the dual graph by one as we join two unconnected dual vertexes in the dual configuration, or we delete an edge which leads to joining two vertexes of the dual that were already connected, so that the number of clusters of the dual remains the same, but so does the number of faces of the original graph. Also, during this deletion process, at each step the number of faces of the original graph is the same as the number of components of the dual graph, so this holds also after the last deletion step. Now since the original and dual configurations are in bijective correspondence, the probability of observing a dual configuration ω_d is the same as the probability of

observing the original configuration ω; also

$$\begin{aligned}
Z_{G,p,q} &= \sum_{\omega\in\Omega} p^{|O(\omega)|}\,(1-p)^{|E|-|O(\omega)|}\,q^{C(\omega)} \\
&= \sum_{\omega_d\in\Omega_d} p^{|E_d|-|O(\omega_d)|}\,(1-p)^{|O(\omega_d)|}\,q^{|V(G)|-|O(\omega)|+F(\omega)-1} \\
&= \sum_{\omega_d\in\Omega_d} p^{|E_d|-|O(\omega_d)|}\,(1-p)^{|O(\omega_d)|}\,q^{|V(G)|-|E_d|+|O(\omega_d)|+C(\omega_d)-1} \\
&= q^{|V(G)|-|E_d|-1} \sum_{\omega_d\in\Omega_d} p^{|E_d|-|O(\omega_d)|}\,((1-p)\,q)^{|O(\omega_d)|}\,q^{C(\omega_d)} \\
&= \frac{(q+p\,(1-q))^{|E_d|}}{(q+p\,(1-q))^{|E_d|}} q^{|V(G)|-|E_d|-1} \sum_{\omega_d\in\Omega_d} p^{|E_d|-|O(\omega_d)|}\,((1-p)\,q)^{|O(\omega_d)|}\,q^{C(\omega_d)} \\
&= (q+p\,(1-q))^{|E_d|}\,q^{|V(G)|-|E_d|-1} \\
&\quad \cdot \sum_{\omega_d\in\Omega_d} \left(\frac{p}{q+p\,(1-q)}\right)^{|E_d|-|O(\omega_d)|} \left(\frac{(1-p)\,q}{q+p\,(1-q)}\right)^{|O(\omega_d)|} q^{C(\omega_d)} \\
&= (q+p\,(1-q))^{|E_d|}\,q^{|V(G)|-|E_d|-1} Z_{G_d,p_d,q_d}.
\end{aligned}$$

Now we get

$$\begin{aligned}
P(\omega_d) &= P_{G,p,q}(\omega) \\
&= \frac{1}{Z_{G,p,q}} p^{|O(\omega)|}\,(1-p)^{|E|-|O(\omega)|}\,q^{C(\omega)} \\
&= \frac{1}{Z_{G,p,q}} p^{|E_d|-|O(\omega_d)|}\,(1-p)^{|O(\omega_d)|}\,q^{|V(G)|-|O(\omega)|+F(\omega)-1} \\
&= \frac{1}{Z_{G,p,q}} p^{|E_d|-|O(\omega_d)|}\,(1-p)^{|O(\omega_d)|}\,q^{|V(G)|-|E_d|+|O(\omega_d)|+C(\omega_d)-1} \\
&= \frac{1}{Z_{G,p,q}} q^{|V(G)|-|E_d|-1} p^{|E_d|-|O(\omega_d)|}\,((1-p)\,q)^{|O(\omega_d)|}\,q^{C(\omega_d)} \\
&= \frac{1}{Z_{G,p,q}} \frac{(q+p\,(1-q))^{|E_d|}}{(q+p\,(1-q))^{|E_d|}} q^{|V(G)|-|E_d|-1} p^{|E_d|-|O(\omega_d)|}\,((1-p)\,q)^{|O(\omega_d)|}\,q^{C(\omega_d)} \\
&= \frac{1}{(q+p\,(1-q))^{|E_d|}\,q^{|V(G)|-|E_d|-1} Z_{G_d,p_d,q_d}} (q+p\,(1-q))^{|E_d|}\,q^{|V(G)|-|E_d|-1} \\
&\quad \cdot \left(\frac{p}{q+p\,(1-q)}\right)^{|E_d|-|O(\omega_d)|} \left(\frac{(1-p)\,q}{q+p\,(1-q)}\right)^{|O(\omega_d)|} q^{C(\omega_d)} \\
&= \frac{1}{Z_{G_d,p_d,q_d}} \left(\frac{p}{q+p\,(1-q)}\right)^{|E_d|-|O(\omega_d)|} \left(\frac{(1-p)\,q}{q+p\,(1-q)}\right)^{|O(\omega_d)|} q^{C(\omega_d)}.
\end{aligned}$$

From the above we see that the dual configuration is distributed like a random cluster configuration with parameters $p_d = \frac{(1-p)q}{q+p(1-q)}$, $q_d = q\,(1-p_d = \frac{p}{q+p(1-q)})$. Note that the self-dual value of p is easily solved for $q \geq 1$ (note

that we must demand $p > 0$):

$$\begin{aligned}
p_d &= \frac{(1-p)\,q}{q + p\,(1-q)} = p \\
&\Leftrightarrow q - pq = pq + p^2 - qp^2 \\
&\Leftrightarrow p^2\,(1-q) + 2pq - q = p^2 - q\,(p-1)^2 = 0 \\
&\Leftrightarrow p^2 = q\,(p-1)^2 \\
&\Rightarrow p = -\sqrt{q}\,(p-1) > 0 \\
&\Leftrightarrow p = \frac{\sqrt{q}}{1+\sqrt{q}}.
\end{aligned}$$

In the above we have utilized only standard duality theory; but as we have warned the reader, we are about to make slight modifications to the concept of duality. Namely we note that given some subgraph of a lattice as our original graph, for example the *n-box* $\Lambda(n) = [-n, n]^2$ in the square lattice (by witch we of course mean the subgraph of the square lattice whose vertexes and edges are contained in the plane domain $[-n, n]^2$), the dual graph of this graph might be such that it is impossible to consider it as a subgraph of a lattice. However we would very much like the dual to be such that it is also a (partial) lattice graph. Hence we modify duality such that we achieve this. So we state that we define the dual of a given lattice subgraph in a case-by-case fashion, such that the dual graph of a lattice graph is a subgraph of the dual lattice. This is the usual method for defining duality in the study of lattice models; so to say it once more, a dual graph of a lattice graph in a study of lattice models is not necessarily the graph-theoretical dual of the graph, but just some suitable dual-like graph that is defined on the dual lattice.

The most important application of duality in two dimensions is the following fact: the cluster of the original lattice is surrounded by a cluster of the dual lattice; this result is sometimes called the *Whitney theorem*. Note specifically that if vertexes x and y do not have a path in the original graph joining them, then there is a *contour* (a circle-path) in the dual lattice such that the other vertex is inside and the other outside this path. From this fact that a cluster is surrounded by a dual cluster it follows that given a box in the square lattice (not necessarily a square-shaped but a rectangular box) then there exists an open crossing from left to right of the box (in an obvious sense) if and only if there does not exist an open crossing of the dual lattice from top to bottom. Of course, the roles of left, right and top, bottom can be reversed. In general, for an event A of the original random cluster model, we define the *dual event* $A_d = \{\omega_d \in \Omega_d \,|\, \omega \in A\}$, and we note that the existence of a horizontal (left to right) crossing and the existence of the dual vertical (top to bottom) crossing are complementary events.

Let us also mention an application of planar duality that we shall apply directly below when we discuss site percolation on the triangular lattice.

Namely, it is obvious that site percolation on the dual lattice is equivalent to face percolation on the original lattice (in *face percolation* one assigns the *open/closed* property to the faces of a graph, with two faces sharing an edge being considered as neighbours when the percolation clusters of neighbouring faces of similar assigned type are formed). Because the triangular lattice is dual to the hexagonal lattice, the site percolation on the triangular lattice corresponds to face percolation on the hexagonal lattice.

7.3. **Russo-Seymour-Welsh estimates for percolation.**

7.3.1. *Smirnov's argument for triangular lattice.* Let us study site percolation on the 2-dimensional *triangular lattice* $\mathbb{T}$ in the critical case, that is, with site-probability $p = \frac{1}{2}$, and give a proof of the box-crossing property through proving the Russo-Seymour-Welsh estimate following *Stanislav Smirnov*'s argument (the author learned the argument from [77]). The argument will rely heavily on the inherent symmetry of the triangular lattice and on the symmetry of the critical case $p = \frac{1}{2}$, that is, the symmetry of equal probability for open and closed vertexes. However the principal ideas of the argument can be adapted to cover the critical percolation models on some other lattices which have sufficient symmetry, like for example, the bond percolation on square lattice. Since the box-crossing property on the square lattice is of great importance due to its generalizability, we shall demonstrate below the application of Smirnov's argument to the square lattice. After establishing box-crossing property for square lattice, one can use a different technique called *star-triangle transformation* to generalize the estimate to a voluminous class of graphs called *isoradial graphs* (that fulfil some conditions). For more information, see [40].

As described above, the triangular lattice is formed by covering the plane with equilateral triangles and taking as lattice vertexes the vertexes of the triangles and as lattice edges the edges of the triangles. We make the arbitrary choice of positioning the lattice such that one of the vertexes of the lattice corresponds to the origin $(0, 0)$ in the 2-dimensional Euclidean coordinates of the plane. By a *rectangle* $R(a, b, c, d)$, a,$b \in \mathbb{Z}$, c,$d \in \frac{\sqrt{3}}{2}\mathbb{Z}$ (we choose the values of c,d like this because the height of an equilateral triangle with side length 1 is $\frac{\sqrt{3}}{2}$) we mean the subgraph of the lattice whose vertexes are contained in the rectangle $[a, b] \times [c, d] \subseteq \mathbb{R}^2$ and that has the edges of the lattice that have both endpoints in the above mentioned set of vertexes. When $a = 0 = c$ and $b \in \mathbb{N}$, $d \in \frac{\sqrt{3}}{2}\mathbb{N}$ we write $R(a, b, c, d) = R(b,d)$. Let $H(a, b)$ be the event that the rectangle $R(a, b)$ is traversed by an open path. By this we mean that a vertex in the "left" boundary of $R(a, b)$ (the *left boundary* of $R(a, b)$ clearly consist of those vertexes whose x-coordinate is 0 or $\frac{1}{2}$; we consider the x-axis to be horizontal) is connected by an open path to a vertex in the "right" boundary of $R(a, b)$ (the *right boundary* of $R(a, b)$ consist clearly of those vertexes whose x-coordinate is a or $a - \frac{1}{2}$). Clearly $a \mapsto P(H(a, b)), a \in \mathbb{N}$ is a non-increasing and $b \mapsto P(H(a, b)), b \in \frac{\sqrt{3}}{2}\mathbb{N}$ is

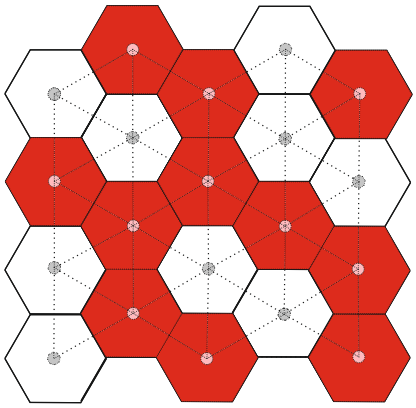


FIGURE 10. The pictorial representation of the site percolation on the triangular lattice. We draw a hexagon around each vertex of the triangular lattice and colour the hexagon red if the corresponding vertex is open and leave it white otherwise. We adopt this pictorial representation solely because it is easier for the eye than just representing the state of vertexes of the triangular lattice by colouring these sites.

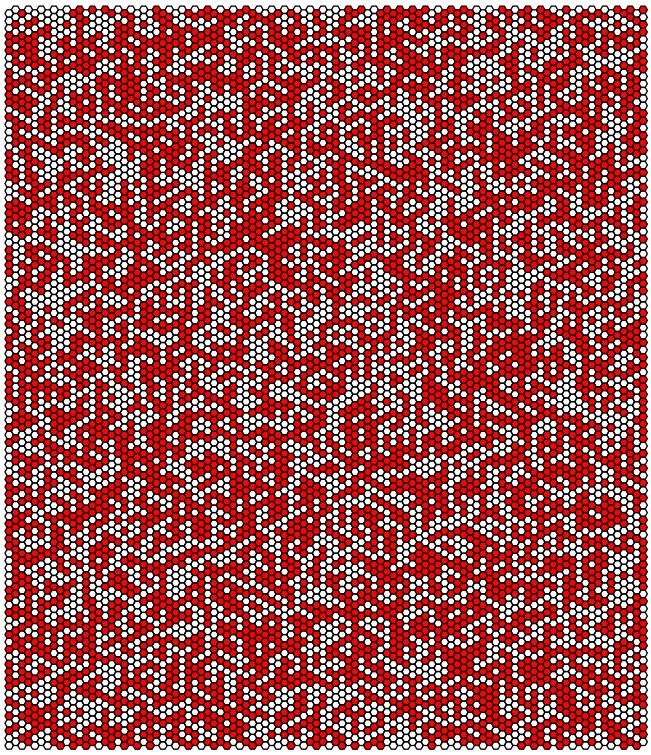


FIGURE 11. A simulation of the critical percolation in triangular lattice in the plane (the figure is from *Michael Kozdron*'s web page and is produced by a simulation written by *Edward Doolittle*).

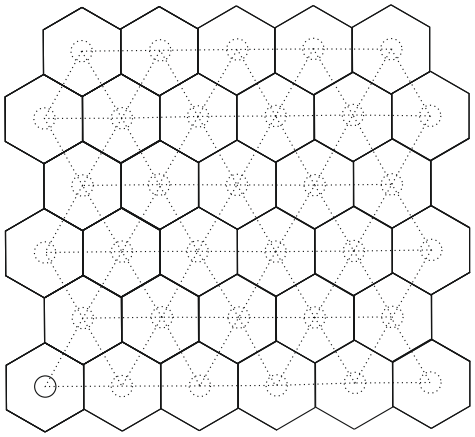


FIGURE 12. Here we have an example on the domain $R(5, 5\frac{\sqrt{3}}{2})$. The vertex corresponding to origin is denoted by a circle with continuous borderline.

a non-decreasing function, and clearly the placement of the rectangle is not relevant, meaning that although we have agreed that $R(a, b)$ has the origin as one corner, all rectangles which have the same number of vertexes in their horizontal and vertical sides have the same crossing probability independent of their position in the plane. Note also that because of symmetry in the lattice, the rectangle can also be rotated $\frac{\pi}{3}$ radians without making any relevant change to the situation (rotation by some other amount of radians may cause troubles). The desired result is the following:

Theorem 7.6. *(*Russo-Seymour-Welsh estimate*) Let* $a \in \mathbb{N}$, $b \in \frac{\sqrt{3}}{2}\mathbb{N}$. *Then*

$$P(H(2a, b)) \geq \frac{P(H(a, b))^2}{4}.$$

Proof. Let l be deterministic a non-self-intersecting (simple) path in the lattice connecting the left side of $R(a, b)$ to the right side of $R(a, b)$. Let us suppose that there exist an open crossing of $R(2a, b)$ (see figure 13); then there must exist an open crossing of $R(a, b)$, and let γ be the "highest" open non-self-intersecting path connecting the left side of $R(a, b)$ to the right side (so γ is a random lattice path). Now if $\gamma = l$, then we see that the vertexes in l must be open and the situation must be such that in the region above l one cannot find a simple path connecting the left side to the right side, but no condition need to be imposed on the states of vertexes that are below l. This means that the event $\{\gamma = l\}$ is independent of the states of vertexes below l. Now denote by l' the path in the lattice formed by mirroring the path l with respect to the line $x = a$; see figure 14. We would like to concatenate the paths l and l' into path $l \star l'$; if l and l' have a common vertex, this is easily done, and if l and l' do not have a common vertex, we introduce the edge between the last vertex of l and the first vertex of l' to the path $l \star l'$. Let C be the connected component of $R(2a, b)$ below $l \star l'$ (we agree that

$l \star l' \not\subset C$) that contains the point $(a, 0)$ (the special case $(a, 0) \in l$, id est, $C = \emptyset$, is handled below). Now we see that geometrically, C is a simply connected set whose boundary is symmetric with respect to line $x = a$. Let us suppose that l does not touch the real axis. Then the boundary of $C \cup l \cup l'$ composes of four parts: l, l', and L, which is the left- and bottom part of the boundary of the part of $R(a, b)$ contained in C and the symmetrical segment L', see figure 15. Note that if l does touch the real axis (but $(a, 0) \notin l$) then the boundary of C can be divided similarly into four parts, but then the parts are: the segment of l that begins from the vertex where the path meets the real axis the last time and ends at the end of l, the symmetrical segment of l', and the part of the real axis to the left of $(a, 0)$ ending at the vertex where real axis meets the above described segment of l, and a symmetrical part of the real axis to the right; by studying these four parts of the boundary instead of the l, l', L, L' described above, the same argument that we present below can be performed.

Now we notice that the probability of an open crossing of C from the outer boundary of l (that is, from those vertexes of C having a neighbouring vertex in l) to L' (note that $L' \subset C$, so a vertex of L' must be present in this crossing) is $\frac{1}{2}$, no matter what the curve l is. This is because if there does not exist an open crossing from the outer boundary of l to L', then the set A of open vertexes of C that are connected to the outer boundary of l with a path consisting of open vertexes (if a vertex of the open boundary of l is closed, it is not included in A) does not intersect L'; since the outer boundary of this set A in C consist necessarily of closed vertexes, and the "upper right" part of this outer boundary of A meets the outer boundary of the path l' (it must meet the outer boundary of l', otherwise the set A a should be connected to L' in the vertical line $x = 2a$) and the outer boundary is connected as a graph (because of the structure of the triangular lattice; note that the outer boundary of A can include vertexes of the outer boundary of l), this set has a closed path connecting the outer boundary of l' and L, for if the outer boundary of A does not meet L, then A should "go around it from below" and meet L' in the horizontal line $y = 0$, see figure 16. On the other hand, we see that we cannot have both an open path connecting the outer boundary of l and L' and a closed path connecting the outer boundary of l' and L, see figure 17. So we have either an open path connecting the outer boundary of l to L' or a closed path connecting the outer boundary of l' to L but not both. Because of the geometrical symmetry of C, we see that an open path connecting the outer boundary of l to L' is just as probable as an open path connecting the outer boundary of l' to L and since $p = \frac{1}{2}$, we see that an open path is just as probable as a closed path; therefore we see that the probability of an open path connecting the outer boundary of l to L' is $\frac{1}{2}$. Let us call this event $A(l)$. Now the events $\{\gamma = l\}$ and $A(l)$ are independent, since $\{\gamma = l\}$ is independent of the states of vertexes below l, and $A(l)$ depends only on the states of those vertexes (the reason we talk about the outer boundaries of l and l' above is that we want this independence to hold).

If we have $\{\gamma = l\}$ and $A(l)$, then there exist an open path from the left side of $R(2a, b)$ to the union of the right side of $R(2a, b)$ and the right half of the bottom of $R(2a, b)$. Call this event A', see figure 18.

When l touches the real axis, a similar argument shows $P(A(l)) = \frac{1}{2}$ and $P(A' \mid \gamma = l) \geq P(A(l))$. If the point $(a, 0)$ is a part of l, then we see that A' happens necessarily (we include the point $(a, 0)$ to both halves of the bottom of $R(2a, b)$). Let us agree that in this case we mean by $A(l)$ the trivial event $A(l) = \Omega$ (Ω is the whole probability space) so that in this case also $P(A' \mid \gamma = l) = 1 \geq P(A(l)) = 1$.

Now, let us notice $A' \subseteq H(a, b) = \cup_l \{\gamma = l\}$. Thus

$$\begin{aligned} P(A') &= \sum_l P(A' \cap \{\gamma = l\}) \\ &= \sum_l P(A' \mid \gamma = l) P(\{\gamma = l\}) \\ &\geq \sum_l P(A(l)) P(\{\gamma = l\}) \\ &\geq \sum_l \frac{1}{2} P(\{\gamma = l\}) \\ &= \frac{P(H(a, b))}{2}. \end{aligned}$$

Now, let us denote by A'' the event that there exist an open path from the right side of $R(2a, b)$ to the union of the left side of $R(2a, b)$ and the left half of the bottom of $R(2a, b)$, see figure 18. By symmetry $P(A'') = P(A')$, and when A'' and A' happen simultaneously, then we have $H(2a, b)$ (remember that the point $(a, 0)$ was chosen to be a part of both halves of the bottom), see figure 19. Since A', A'' are both increasing events, we can conclude from the Harris' inequality:

$$P(H(2a, b)) \geq P(A' \cap A'') \geq P(A')P(A'') \geq \frac{P(H(a, b))^2}{4}.$$

□

This result has a major corollary as follows: since the rhombus that has two of its sides horizontal and the other two sides in $\frac{\pi}{3}$-angle with the horizontal, that is, parallel to the non-horizontal axis of triangular lattice, and has side-length $2n$, contains a rectangle that has side-lengths n horizontal and $2n \cdot \frac{\sqrt{3}}{2} = n\sqrt{3}$ vertical, this means that a horizontal crossing of the above mentioned rhombus contains a horizontal crossing of the rectangle, see figure 20a. Since a rhombus either has an open horizontal crossing or, if that is not the case, then the outer boundary of the set A of open vertexes of the rhombus that can be linked to the left boundary of the rhombus by an open path that is inside the rhombus is necessary closed (consist of closed vertexes) and the part of it that is contained in the rhombus forms a continuous "wall" connecting the upper boundary of the rhombus to the lower boundary (if this would not hold, then the set A could be extended to meet

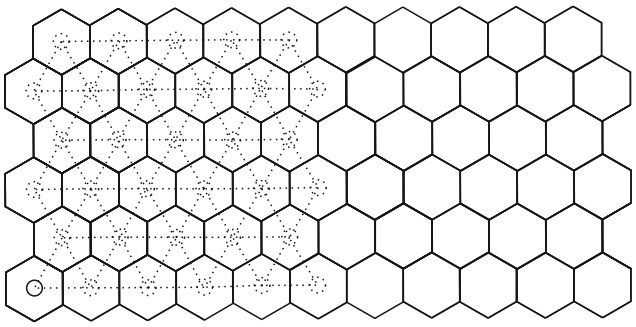


FIGURE 13. Here we have an example of the domain $R(10, 5\frac{\sqrt{3}}{2})$. The vertex corresponding to origin is denoted by a circle with continuous borderline.

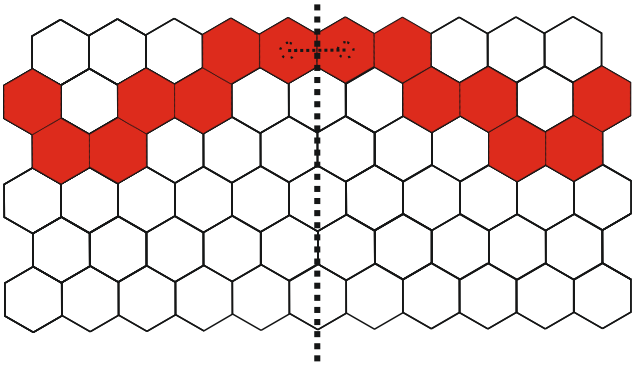


FIGURE 14. Here we have an example of a possible uppermost path l and its mirror image l' with respect to line $x = 5$. Note the extra edge we attach to the concatenation of l and l'.

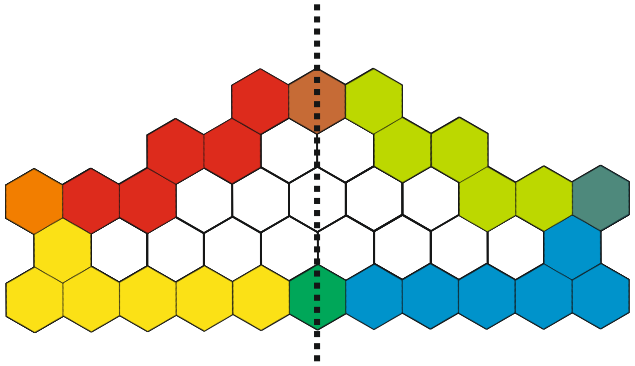


FIGURE 15. Here we have an example of a possible domain C. The colours are as follows: red is for the outer boundary of l, yellow is for L, blue is for L' and green is for the outer boundary of l'; vertexes belonging to two of these sets of vertexes are coloured with mixed color.

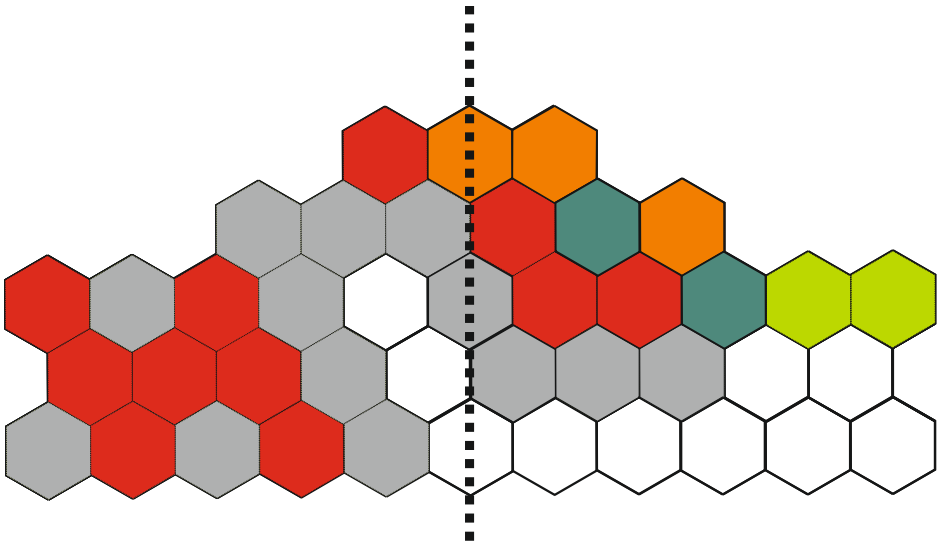


FIGURE 16. Here we have an example of a situation where there is no open (red) crossing from the outer boundary of l to L' in domain C. Note that in this situation there must be a closed crossing from the outer boundary of l' (light green, with orange open and dark green closed vertexes) to L in C, as the grey (the dark green) set of closed vertexes, the outer boundary of the set A, demonstrates.

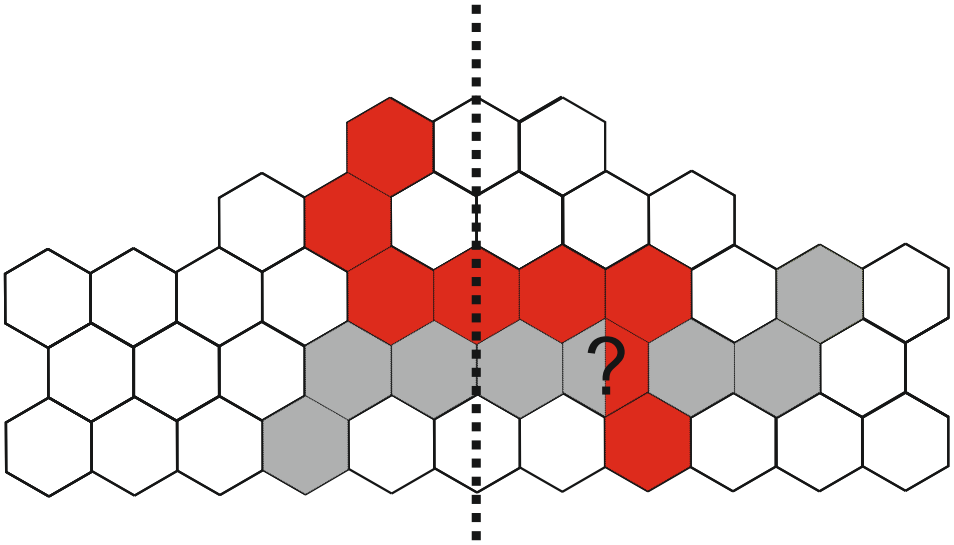


FIGURE 17. Here we have an example of a possible open crossing from the outer boundary of l to L' in domain C. Note that in this situation there cannot be a closed crossing from the outer boundary of l' to L in C (see the hexagon marked with a question mark).

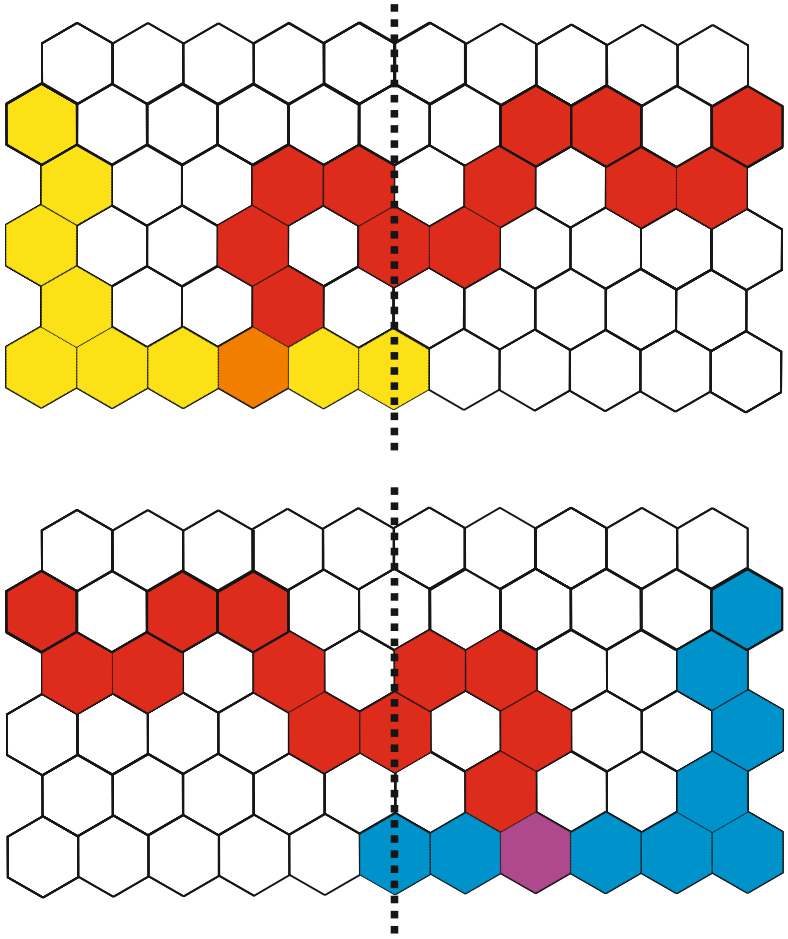


FIGURE 18. Here we have an example of the events A' (upper figure) and A'' (lower figure).

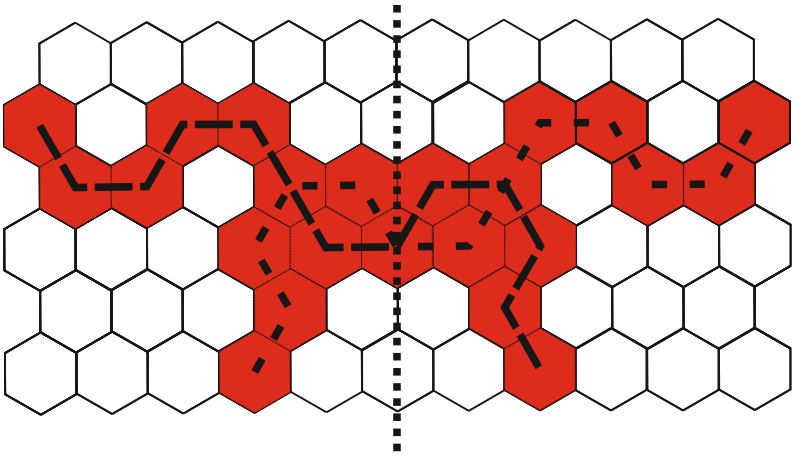
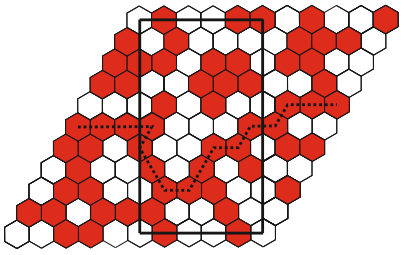
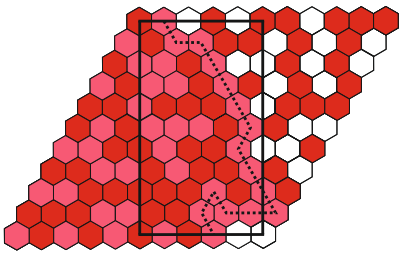


FIGURE 19. Here we have an example of the event $A' \cap A''$.



(A) Here we see how a rectangle can be fitted inside a rhombus and how a horizontal crossing of the rhombus induces a horizontal crossing of the rectangle.



(B) Here we have see that there exist either an open (red) horizontal crossing of the rhombus or a closed (pink) vertical crossing of the rhombus.

with the right boundary of the rhombus, thus forming an open crossing, which is a contradiction with the assumption) and thus this boundary contains a closed vertical crossing as in figure 20b. So the rhombus has either an open horizontal or closed vertical crossing but not both, and since the rhombus is symmetric in the sense that a horizontal crossing must have the same probability as a vertical crossing, and since $p = \frac{1}{2}$, meaning that the probability is preserved when we change open vertexes to closed ones and vice versa, we see that open horizontal and closed vertical crossings have the same probability, so we have that the rhombus has an open crossing with probability $\frac{1}{2}$. So the probability of an open crossing of the rectangle is greater or equal to $\frac{1}{2}$; since the probability of an open crossing is the same for all rectangles with the same sides, we get that $P(H(n, n\sqrt{3})) \geq \frac{1}{2}$. Now, by using this and the above theorem and the non-increasing and non-decreasing properties of the functions $a \mapsto P(H(a,b))$, $b \mapsto P(H(a,b))$, we get that

Corollary 7.1. *For $k > 0$ there exist $a_k > 0$ such that for every $n \in \mathbb{N}$ such that $R(kn, n)$ is not the empty graph the following holds:*

$$P(H(kn, n)) \geq a_k > 0.$$

In the above, the value of the bounding constant is not important; what is important is that the probability is uniformly (in n) bounded away from zero. The above leads us to the following result, that is also called *Russo-Seymour-Welsh estimate*.

Let $\Lambda(n) = \{$vertexes of triangular lattice that are of graph distance less or equal to n from the origin$\}$ be the "*n-circle*" and let $A_n = \Lambda(2^{n+1}) \setminus \Lambda(2^n)$ be the "*n-annulus*". Let C_n be the event that there exist an open loop surrounding the origin that is contained in the annulus A_n. We claim

Theorem 7.7. *(*Russo-Seymour-Welsh estimate*) There exist $c > 0$ such that for all $n \in \mathbb{N}$:*

$$P(C_n) \geq c > 0.$$

Proof. The idea of the proof follows a general technique known as "*block argument*". Here we separate the area of the lattice under study into rectangular areas, and since we can approximate the probability of an open crossing of these rectangles, this gives us a way of studying the probability that we are interested in. For example, the annulus A_n can be divided into six rectangles as in the figure 21; now since these rectangles overlap in their ends, we see that if every rectangle is traversed in the long-side-direction by an open path, then these paths together form the open cycle that we want in the event C_n. Since the probability of each rectangle being traversed in the long direction by an open path can be approximated from below by a constant $a > 0$ according to the above theorem, and since the event of all six rectangles being traversed by open paths is obviously an increasing event, we can use Harris' inequality to find that

$$P(C_n) \geq P(\{\text{All rectangles are traversed by an open path}\}) \geq a^6 > 0.$$

□

As a final remark let us generalize the acquired results a bit. For above we have only considered the case $p = \frac{1}{2}$, also the critical case for planar triangular lattice. But we actually see that we can generalize the box-crossing property obtained above for the supercritical case $p > \frac{1}{2}$ by using a technique of coupling different percolation models. Namely, let us view the percolation model on the lattice $\mathbb{T} = (V(\mathbb{T}), E(\mathbb{T}))$ as an infinite collection of identically and independently distributed random variables X_v, $v \in V(\mathbb{T})$ which have even distribution on the interval $[0, 1]$, and given the parameter $p \in [0, 1]$, let us call those vertexes $v \in V(\mathbb{T})$ that have (in configuration $\omega \in \Omega$) $X_v(\omega) \leq p$ *open*, and the others *closed*. Now we see that each vertex is open with probability p independent of all the other vertexes, and closed otherwise; hence our model is the vertex percolation. Now we see that for any configuration $\omega \in \Omega$, if $p' > p$ ($p, p' \in [0, 1]$) then those vertexes that are open in the percolation with parameter p are also open with parameter p'; hence we see that increasing events (those events that benefit from more open vertexes) are more likely to happen in a model with bigger p parameter, as is intuitive. Hence when $p > \frac{1}{2}$, we can say that the probability of existence of an open horizontal crossing of a rectangle or an open loop around the origin is at least as big as in the case $p = \frac{1}{2}$; hence the lower bounds we have provided to these probabilities above hold also for $p > \frac{1}{2}$, that is, these probabilities are uniformly in the size of the rectangle and loop bounded away from zero also in the case $p > \frac{1}{2}$, id est, the box-crossing property holds also in the case $p > \frac{1}{2}$.

7.3.2. *Smirnov's argument for square lattice.* Let us then consider bond percolation on the square lattice $\mathbb{L}^2$. We shall prove the box-crossing property through Russo-Seymour-Welsh estimates using a variant of the Smirnov's argument presented above for the triangular lattice. This argument was presented to the author by *Antti Kemppainen*. The importance of the proof for box-crossing property for the square lattice stems from the above-mentioned

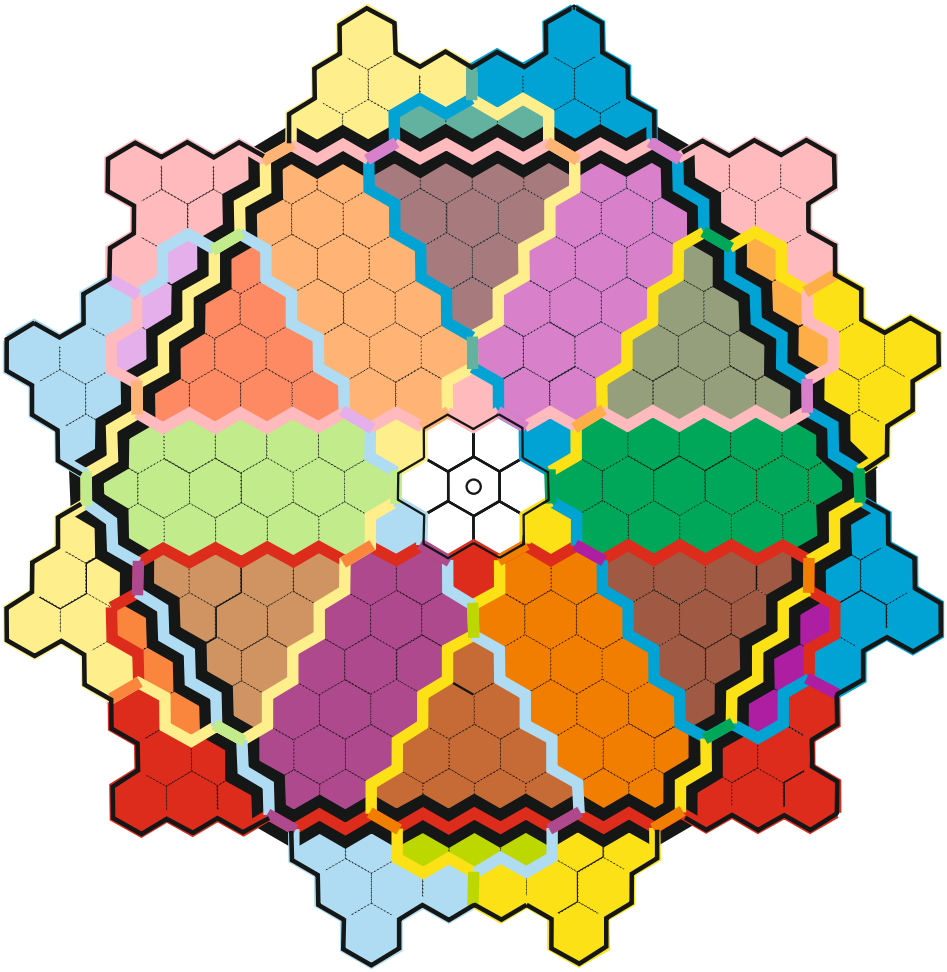


FIGURE 21. The annulus being divided into six rectangles. The colours of the rectangles are blue, pink, light yellow, light blue, red and yellow (in cyclic order); the colors are chosen such that a color and its light version are always complementary.

fact that the result is generalizable to a wide class of graphs, the isoradial graphs (that are in a sense the most general class of graphs that is sensible to study). The generalizability is proved in [40]; note that they call the property that the probability of an open crossing of a topological (that is, up to some transform) rectangle is bounded away form zero the *box-crossing property* (so they are slightly more general than we), and they call the *Russo-Seymour-Welsh lemma* the fact that the bond percolation on the square lattice with edge-probability $p = \frac{1}{2}$ has the box-crossing property.

We study the case where the percolation model is critical, id est, we take $p = \frac{1}{2}$ (it is not trivial that this is the critical case but it is; this can be seen by considering the self-duality of the square lattice percolation, see

the subsubsection on planar duality above). By a *rectangle* $R(a,b,c,d)$, $a,b,c,d \in \mathbb{Z}$ we mean the subgraph of the lattice whose vertexes are in the plane-domain $[a,b] \times [c,d]$. Because of the translational symmetry of the percolation model (the location in the plane does not affect the probabilities) we assume that our rectangle has the origin as lower left corner, id est, the rectangle is $R(0,b,0,d) = R(b,d)$, $b,d \in \mathbb{N}$, and we denote the event that the rectangle is *horizontally crossed by an open path* by $H(b,d)$, meaning that there exist an open path from $\{0\} \times [0,d]$ to $\{b\} \times [0,d]$. Again we note the functions $a \mapsto P(H(a,b))$, $b \in \mathbb{N}$ and $b \mapsto P(H(a,b))$, $a \in \mathbb{N}$ are non-increasing and non-decreasing, respectively.

We shall first study $\frac{\pi}{4}$-rotated rectangles and prove the Russo-Seymour-Welsh estimate for them and focus on the "straight" rectangles whose sides are parallel to coordinate axes after that. This is because studying rotated rectangles allows us to utilize the symmetry between original and dual lattices to greater extend than would be possible if we focused only on the "straight" rectangles; ultimately this is due to the fact that the square lattice and dual square lattice (that is just a shifted square lattice) have a common direction of symmetry that is easily accessible in the case of rotated rectangles and not so easily in the case of "straight" rectangles. Let us note explicitly that in the case of the square lattice we need to study both the lattice and its dual whereas in the case of the triangular lattice we need not refer to the dual lattice. This is due to the fact that in the triangular lattice the dual lattice is "inside" the original lattice, meaning that the dual contours encircling the original lattice's percolation components manifest themselves also in the original lattice; this is not true in the case of square lattice, and this is why we need to bring the dual lattice (and with it, some minor technical details) to the analysis. We define the *rotated* $a \times b$ *rectangle* $RO(a,b)$, $a,b \in \mathbb{N}$ to be the rectangle having a columns of diagonal squares and b rows of diagonal squares as seen by rotating the viewpoint of the observer by $\frac{\pi}{4}$; see the figure 22 where we portray a 5×10 rotated rectangle. By the *left* and *right borders of the rotated rectangle* we mean the outermost vertexes of the rectangle as viewed by changing the orientation of the observer by $\frac{\pi}{4}$, see figure 22 where the border vertexes are denoted by balls. By a (*horizontal*) *crossing* of the rectangle we mean an open path connecting two vertexes of different borders; we denote this event (that such a path exist) $CRO(a,b)$. In connection with a rotated rectangle $\mathrm{RO}(a,b)$ we define its *dual domain* in the dual lattice as the domain $\mathrm{RO}(a,b)+(\frac{1}{2},\frac{1}{2})$, see figure 23. Note that the dual of rotated rectangle is not its graph-theoretic dual but a "dual-like" object defined on the dual lattice.

We note that since we have $p = \frac{1}{2}$, the percolation on the dual lattice is obviously symmetric to the percolation on the original lattice (the dual and the original lattice being isomorphic). For last we note that the lattice and hence the whole set-up is symmetric with respect to $\frac{\pi}{2}$-rotation.

Theorem 7.8. *(*Russo-Seymour-Welsh estimate*) For $a \in \mathbb{N}$, $b \in \mathbb{N}$ we have:*

$$P(CRO(2a, b)) \geq \frac{P(CRO(a, b))^2}{16}.$$

Proof. As said above, the following argument is similar to the above presented argument for the triangular lattice. First we note that in order for there to exist an open horizontal crossing of RO$(2a, b)$, there must be an open crossing of RO(a, b). Let l be a deterministic path in the lattice joining the left and right borders of RO(a, b) and let γ be the highest non-self-intersecting (random) path crossing RO(a, b) in the observed percolation configuration. Now if we have $\gamma = l$ we see as above that the edges of l must be open and the situation in the part of the domain RO(a, b) above l must be such that there is no open crossing of RO(a, b) in that part of RO(a, b); and again the event $\gamma = l$ is independent of the states of the edges below l. Now denote by l' the lattice path formed by mirroring the path l with respect to the black line of figure 24. Note that the line in question goes between the original and dual lattice such that l' is a dual lattice path. Now note that l' is in the same position in the dual domain of the rotated rectangle as l is in the original domain; namely, l' connects the left and right borders of the dual rectangle's "half-rectangle". Note that since $p = \frac{1}{2}$, the dual percolation and percolation are equivalent and since the rotated rectangle and its dual domain are completely symmetric (the location and the "direction" do not affect percolation), their respective percolation processes are completely equivalent.

Now using the paths l and l', let us define (an analog of the domain C above) two domains C and C' such that C is a the part of RO(a, b) (so C is part of the original lattice) below l and l' and C' is the part of the dual of RO(a, b) (so C' is part of the dual lattice) below l and l'; look at figure 25. Now we define two special sets of boundary vertexes of C (and C'), namely the *left upper* (*right upper*) *corner* and *left* (*right*) *lower part* in case of C (C') which we denote in the figure 25 by blue (red) and yellow (green) balls, respectively. We shall study crossings of C (C') between these sets. Note that these kinds of crossings, connecting vertexes of these sets, need not include edges between vertexes of the same color because these edges are unnecessary for these crossings to take place, so below when we say "crossing" we mean such a lattice path that does not include edges that connect two vertexes of the same color. As we see from the figure 25, we cannot have both an open crossing of C from blue to yellow and a dual open crossing of C' from red to green vertexes at the same time, for these crossing would necessarily intersect, which is impossible. On the other hand, as is depicted in figure 26 we see that if we do not have an open crossing of C from the blue vertexes to the yellow vertexes, then the dual contour encircling the connected open component (of the original lattice) of the yellow vertexes forms a path that guarantees the we have a dual open crossing of C' from the red to the green vertexes. Also the events "exists

an open crossing of C from the blue to the yellow vertexes" and "exists a dual open crossing of C' from the red to the green vertexes" partition the probability space; by symmetry, these events must be of equal probability. Hence their probability must be $\frac{1}{2}$ no matter what the path l was.

Let us now call the event "exists an open crossing of C from the blue to the yellow vertexes" $A(l)$. We note that the events $\gamma = l$ and $A(l)$ are independents since $\gamma = l$ depends only on the states of edges above and at l, whereas $A(l)$ depends only on the state of edges below l; remember what was said above about crossings using edges between vertexes of the same color. Now we see that if we have $\gamma = l$ and $A(l)$, we do not necessarily have an open crossing from the left boundary of $\mathrm{RO}(2a, b)$ to the right part of the bottom of $\mathrm{RO}(2a, b)$ - let us call this event A' - for there is a little technical hitch portrayed in figure 27. We see that if the crossing in the event $A(l)$ is like the crossing p in the figure, then there is no problem in the sense that A' happens since the edges of $\gamma = l$ are open (gray background in figure 27). On the other hand, if the crossing in the event $A(l)$ is like the crossing p' in the figure 27, then A' does not happen, because if the orange edge in figure is closed, there is no open crossing like A' requires. However, the probability of the orange edge being open is $\frac{1}{2}$ and it is independent of all things; hence we can conclude that by introducing a factor $\frac{1}{2}$ in the probability $P(A(l) \cap \{\gamma = l\})$ we have

$$\begin{aligned}
P(A') &= \sum_l P(A' \cap \{\gamma = l\}) \\
&\geq \sum_l P(A(l) \cap \{\gamma = l\} \cap \{\text{end of crossing open if needed}\}) \\
&\geq \sum_l \frac{1}{2} P(A(l)) P(\gamma = l) \\
&= \sum_l \frac{1}{2} \cdot \frac{1}{2} P(\gamma = l) \\
&= \frac{1}{4} \sum_l P(\gamma = l) \\
&= \frac{1}{4} P(\mathrm{CRO}(a, b)).
\end{aligned}$$

Now let us define a symmetric event A'' as the event that there exist an open crossing from the right boundary of $\mathrm{RO}(2a, b)$ to the left part of the bottom of $\mathrm{RO}(2a, b)$. By symmetry $P(A'') = P(A')$ and obviously (see figure 28) by Harris' inequality

$$P(\mathrm{CRO}(2a, b)) \geq P(A' \cap A'') \geq P(A')P(A'') \geq \frac{1}{16} P(\mathrm{CRO}(a, b))^2.$$

□

Now we prove that the probability of an open crossing of a rotated square $\mathrm{RO}(a, a)$ is $\frac{1}{2}$. First, as we can see in the figure 29, either there is an open

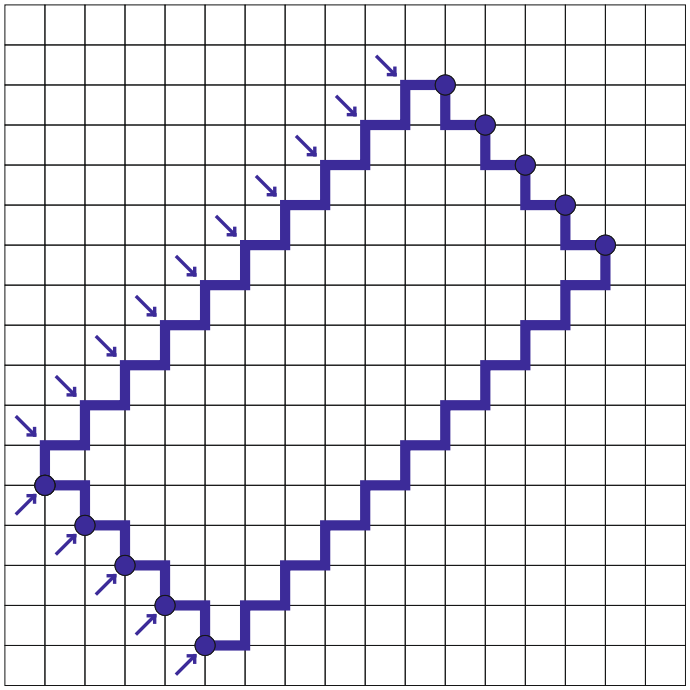


FIGURE 22. The domain RO(10, 5).

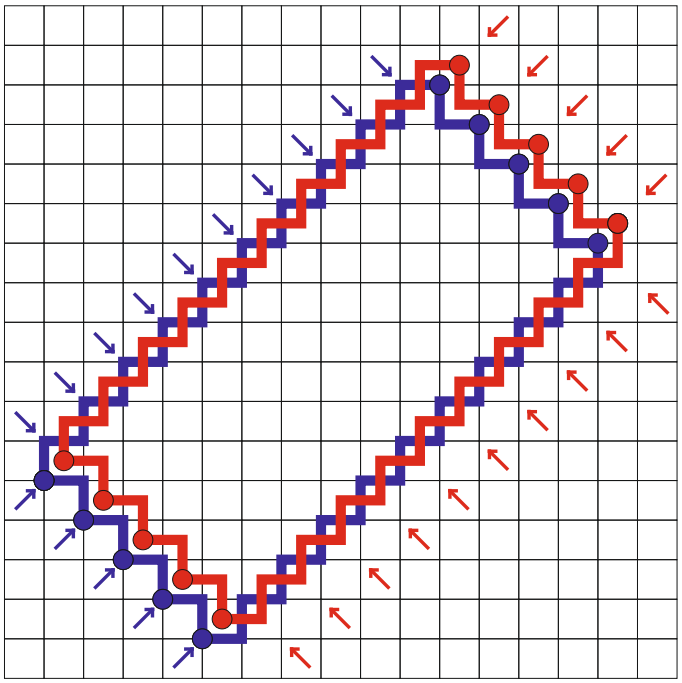


FIGURE 23. The domain RO(10, 5) in blue in the original lattice and its dual domain in red in the dual lattice.

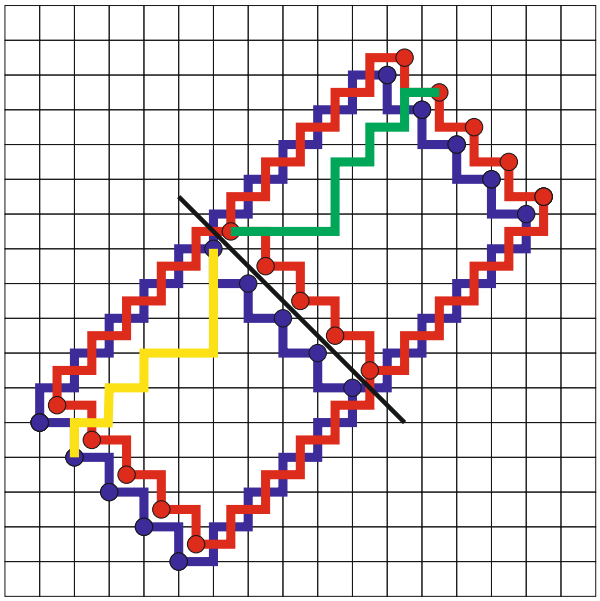


FIGURE 24. The highest path crossing RO(5, 5) included in a crossing of RO(10, 5), l, being reflected with respect to the black line to form a path of the dual lattice l'. Note that l' crosses a domain similar to RO(5, 5) in the dual lattice.

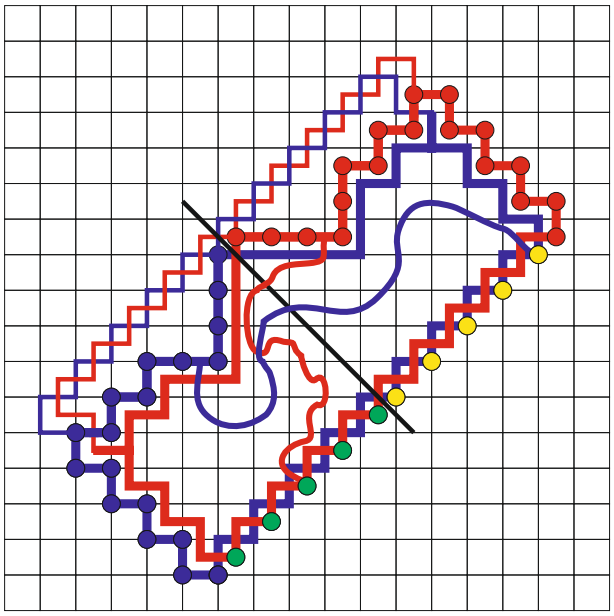


FIGURE 25. The domains C of the original and C' of the dual lattice below the paths l and l'. As is seen, the crossing of the original lattice from the vertexes marked with blue balls to the vertexes marked with yellow balls must intersect a crossing of the dual lattice the vertexes marked with red balls to the vertexes marked with green balls.

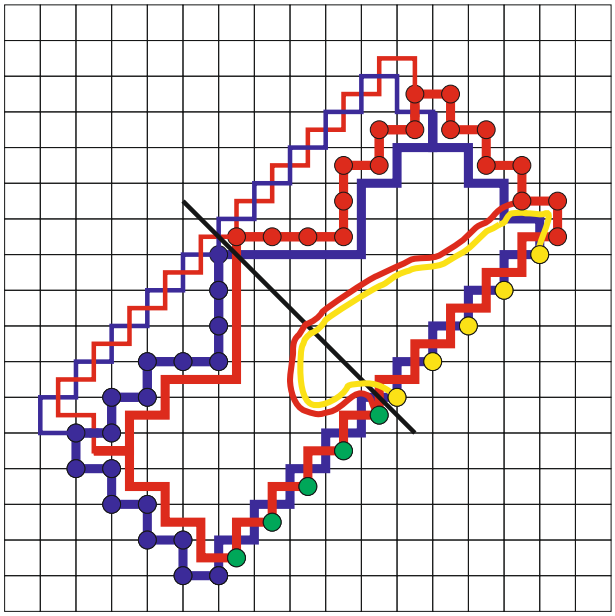


FIGURE 26. Here we see that the boundary curve in the dual lattice of the open connected component of the yellow ball vertexes in the original lattice must connect the red and green vertexes in the dual lattice.

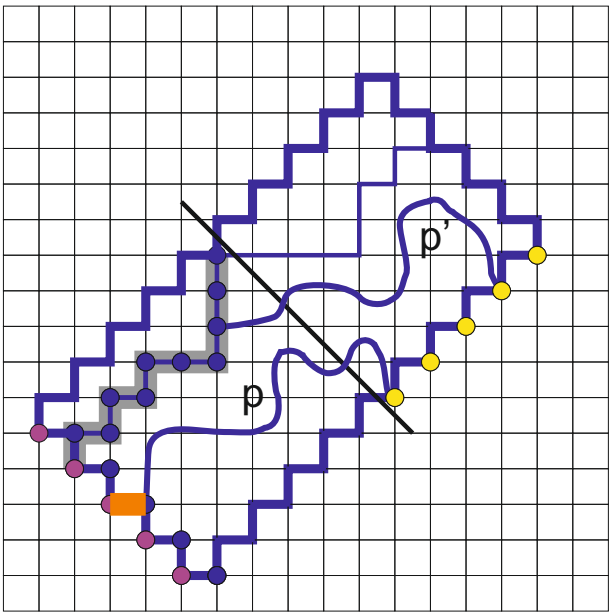


FIGURE 27. Here we see the problem that arises from the fact that the blue vertexes of figure 25 are not completely the same as the boundary vertexes in figure 22. This costs us a multiplicative factor $\frac{1}{2}$.

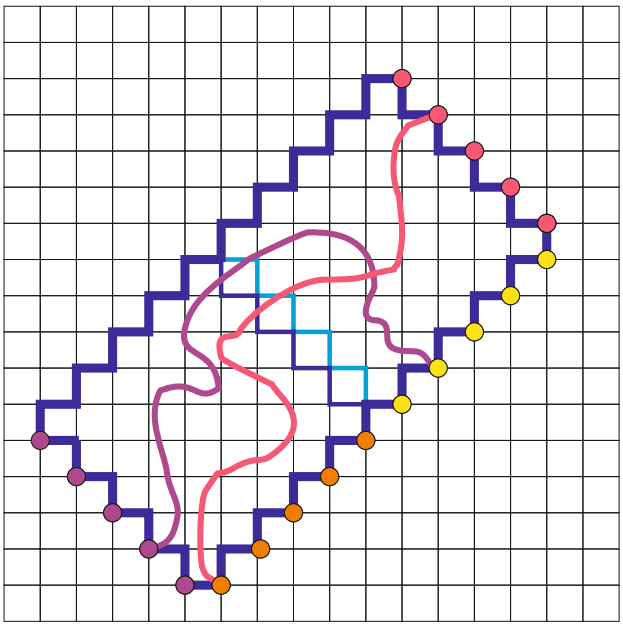


FIGURE 28. Here we see $A' \cap A'' \subset \mathrm{CRO}(2a, b)$.

horizontal crossing of the domain in the original lattice (in blue in figure 29, with crossing between blue vertexes) or there is a dual open vertical crossing of the dual domain in the dual lattice (red in the figure 29, with crossing between red vertexes), since if there is no open horizontal crossing, the dual contour bordering the open connected cluster (of the original lattice) of the left border of the original lattice domain (cluster is yellow in figure 29, and the dual contour is in red) forms an open vertical crossing of the dual domain, and it is clear that it is not possible to have a horizontal open crossing of the original and a vertical dual open crossing of the dual domain simultaneously since this would lead to an intersection between open crossing and a dual open crossing (note how the red and blue crossing intersect in figure 29). Now looking into the dual domain in figure 29 more closely in figure 30 we see that actually the edges marked with orange and the vertexes marked with purple are irrelevant with respect to whether or not there exists a dual open crossing of the domain between the boundaries of red vertexes in the figure 29, for if such an open crossing exists, there necessarily is an open crossing between the red vertex sets of figure 30, and vice versa if there is an open crossing between the red vertexes of figure 30, the there of course is an open crossing between the red vertex boundaries of figure 29. So we can drop the edges marked orange and the vertexes marked purple in figure 30 from the dual domain of figure 29.

Now let us study figure 31. Here we have defined yet another domain in the dual lattice, denoted by yellow line and yellow vertexes in figure 31. We see that this domain is a $\mathrm{RO}(a, a)$-rectangle. Since the dual percolation and

original percolation are equivalent since we have $p = \frac{1}{2}$, the probability of an open crossing of the yellow domain is the same as the probability of an open crossing of the original blue domain of figure 29 (we also appeal to symmetry of the lattices here). Now let ω be a configuration of the yellow domain. We map it to a configuration ω' defined by translating the states of those edges in the yellow domain below the red domain to the edges of the red domain above the yellow domain (those edges are marked orange in figure 31, and the edges to which they map to are denoted purple; the edges are just translated some steps of diagonal squares to upper left direction). It is clear that this mapping preserves probability, id est, it is just as likely for ω to appear in yellow domain as it is for ω' to appear in the red domain; furthermore this mapping between configurations is a bijection. Also we see that there is an open crossing of the yellow domain in ω if and only if there is an open crossing of the red domain in ω' (green open path and light blue open path in figure 31). So now we see that the probability of an open crossing of the yellow and red domains of figure 31 are equal. Hence we see that the the probabilities of open crossings of the red and blue domains of figure 29 are equal. By above we see this probability must be $\frac{1}{2}$.

Hence the probability of an open horizontal crossing of the rotated square is $\frac{1}{2}$. This combined with the above theorem and the above note the monotonicity of the probability of horizontal open crossing with respect to width and length parameters of the rotated rectangle leads us to deduce (as in the case of the triangular lattice percolation) that the probability of an open horizontal crossing of a rotated rectangle is bounded away from zero uniformly in the size (but not in the width-to-length-ratio) of the rotated rectangle.

Let us now focus on the "straight" rectangle. We use a similar "block argument" as we have before to deduce that the probability of an open horizontal crossing of a "straight" rectangle is bounded away from zero uniformly in the size of the rectangle, but the width-to-length ratio of rectangle can have an effect to this probability. Namely, see figure 32; one can place many rotated rectangles in the "straight" rectangle such that if there is an open crossing in every rotated rectangle, the there is an open crossing in the straight rectangle; since the number of rotated rectangles needed depends only on the width-to-length ratio of the "straight" rectangle and not on the size of the rectangle, as the rotated rectangles can be scaled to match the size of the "straight" rectangle, we have by the above box-crossing property and by Harris' inequality, that the probability of an open crossing of the "straight" rectangle is bounded from below by the probability of an open crossing of the rotated rectangle (which is by symmetry the same for all rotated rectangles since they are similar), which is non-zero, raised to appropriate power (the number of rotated rectangles), which means the probability of an open crossing of the "straight" rectangle is bounded away from zero uniformly in the size of the rectangle.

Also, when we consider an annulus (of the graph metric) in the square lattice, we see that by applying the "block argument" and Harris' inequality

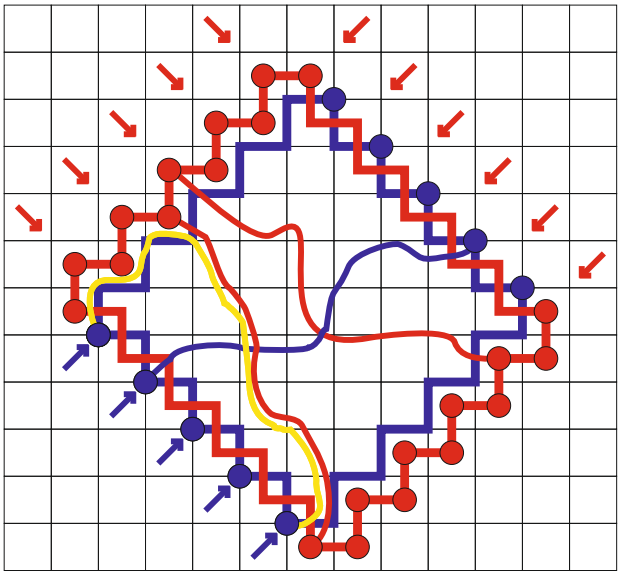


FIGURE 29. Here we see that in our rotated square domain, we have either an open (blue) crossing connecting blue vertexes of different sides of the square or a dual open (red) crossing connecting the red vertex sets of different sides, but not both. Also we necessarily have one of the above crossings as the yellow and red lines depict, the yellow being the hypothetical boundary of the open component of the blue vertexes on the left side and the red line being the dual-open boundary of that component.

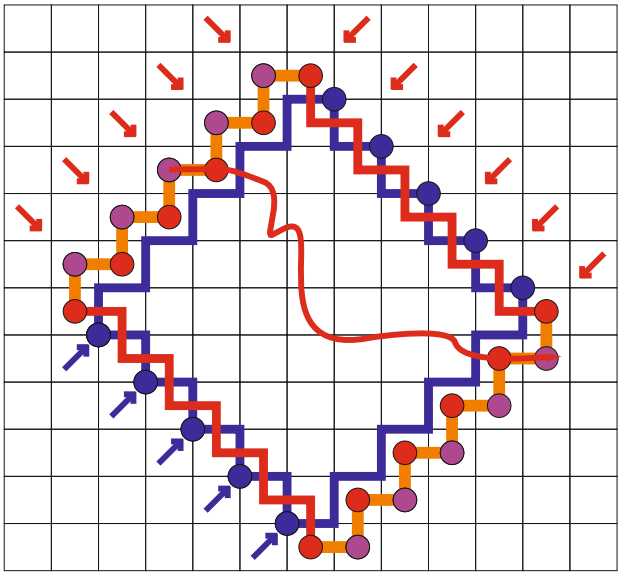


FIGURE 30. Here we see that a red crossing of the figure 29 actually only needs to connect two red vertexes of this figure, the purple vertexes being "useless" in making a dual-open crossing as in figure 29.

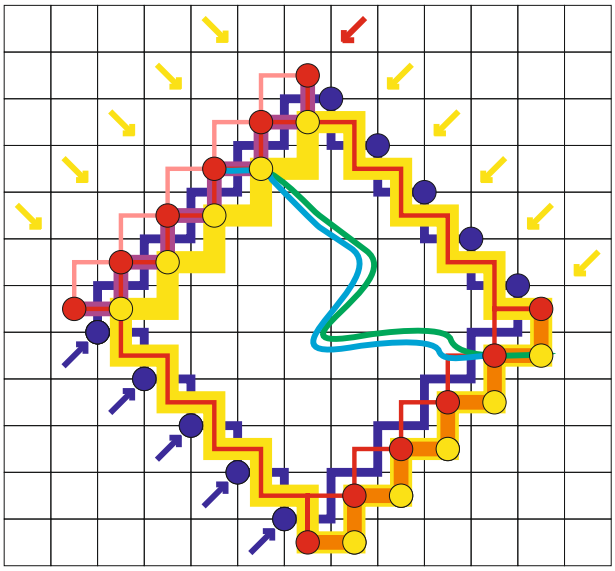


FIGURE 31. Here we define yet another auxiliary domain, marked with thick yellow line. We study crossings from the yellow vertexes on the top to yellow vertexes in the bottom inside this domain, like the green crossing in figure. Obviously the yellow domain is completely symmetric to the original blue domain in this "crossing-sense". Now we make a coupling between configurations of the yellow and of the red domains simply by translating the states of the edges marked in orange to the edges marked in purple one-to-one. Then we map the green crossings into light blue crossings of the red domain. But now, as noted in the caption of figure 30, the crossings of the red domain are actually precisely the crossings of the red domain of figure 29 (the pink line). So we see that in figure 29 the horizontal crossing of the blue domain is equally likely as the vertical crossing of the red.

together with our results on the rotated rectangles we see that the probability of the existence of an open circle inside the annulus is also bounded away from zero uniformly in the size (but not in the width) of the annulus, see figure 33.

Also our above (in the triangular percolation subsubsection) note that the probabilities concerning existence of open crossings satisfy the same lower bounds as in the case $p = \frac{1}{2}$ also in the case $p > \frac{1}{2}$ holds in the square lattice also with the same argument.

7.4. **Box-crossing property of the random cluster model.** We will prove *box-crossing property* of the random cluster model in square 2-dimensional lattice with parameters $p = \frac{\sqrt{2}}{1+\sqrt{2}}$ (the so-called *self dual* value of parameter)

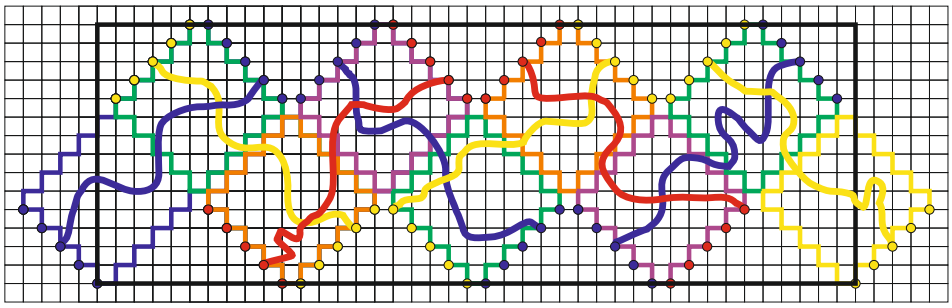


FIGURE 32. In the large straight rectangle one can place many small rotated rectangles such that if all the rotated rectangles are crossed by an open path, then the large straight rectangle is crossed by an open path. The number of rotated rectangles needed to do this does not depend on the size of the large rectangle but only of the aspect ratio of the large rectangle.

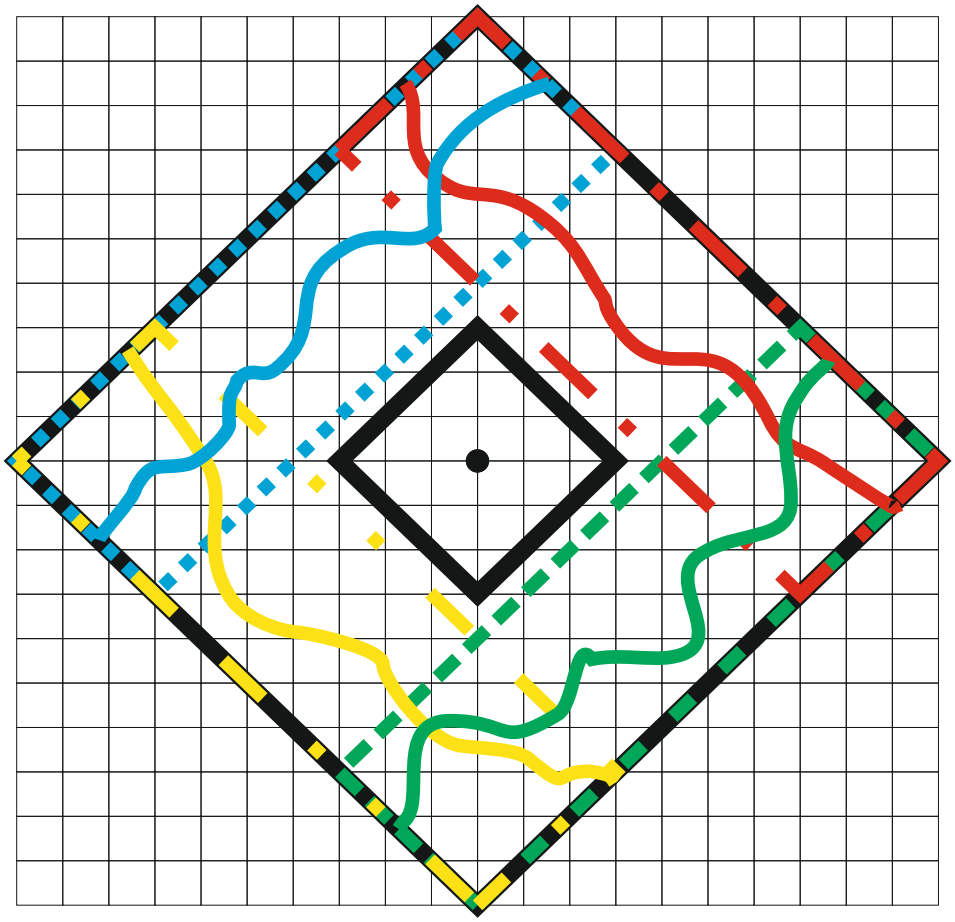


FIGURE 33. The annulus being divided into four (red, blue, yellow, green) rectangles.

and $q = 2$. With parameter $q = 2$, the random cluster model can be interpreted as a geometric representation of the Ising model via Edwards-Sokal coupling. This is why the random cluster model with these parameter values is called the *FK Ising model*. With parameter $q = 2$ and lattice $\mathbb{L}^2$, the FK Ising model features a phase transition at the *critical* parameter value $p_c = \frac{\sqrt{2}}{1+\sqrt{2}}$; for $p < p_c$, the *sub-critical* case, there is (almost surely) no infinite cluster and for $p > p_c$, the *supercritical* case, there is (almost surely) an unique infinite cluster. Because of this choice of parameter p (choosing the critical, self dual value), our model of interest is called the *critical FK Ising model*.

We will use the argument presented by *Hugo Duminil-Copin*, *Clément Hongler* and *Pierre Nolin* in [17], and the argument uses a so-called preholomorphic fermionic observable introduced by Stanislav Smirnov [72]. We shall discuss the square lattice only, but some of the arguments we present can be extended to isoradial graphs following [15]. Also our arguments can possibly be generalized in a manner similar to [40] where the box-crossing property of normal bond percolation, not random cluster percolation, is generalized from the square lattice to isoradial lattices (with some properties). Let us also mention that different approaches than the one we are about to present to prove the box-crossing property have been proposed; in [15], *Dmitry Chelkak* and *Stanislav Smirnov* show how one can compute explicit crossing probabilities in the scaling limit, and in [12] *Federico Camia* and *Charles Michael Newman* propose a proof of the box-crossing property that bases itself on the result announced in [15]: the full collection of interfaces (between zones of +- and −-spins) of the Ising model at criticality converges to so-called *conformal loop ensemble CLE(3)* at scaling limit; the proof proposed also uses the interpretation of CLE(3) as *Brownian loop soup* [76]. However, these proposed proofs are not as general as the proof that we are about to present, for the proposed proofs consider only certain boundary conditions; also the proposed proofs require a notably heavier mathematical machinery than the proof that follows.

We call the subgraph of $\mathbb{L}^2$ contained in $[0, a] \times [0, b] \subset \mathbb{R}^2$, $a, b \in]0, \infty[$ the *rectangle* $R(a, b)$. We denote $V(a, b)$ the event that $R(a, b)$ is vertically traversed by an *open path*, id est, there is a path consisting only of open edges from $[0, a] \times \{0\}$ to $[0, a] \times \{b\}$. We prove:

Theorem 7.9. *Let $0 < \beta < \infty$. There exist two constants $0 < c_1 < c_2 < 1$, depending only on β, such that for all $n \in \mathbb{N}$ holds:*

$$c_1 \leq P^{\xi}_{R(\beta n, n)}(V(\beta n, n)) \leq c_2$$

for all boundary conditions ξ, where the $P^{\xi}_{R(\beta n, n)}$ denote the random cluster measure in the rectangle $R(\beta n, n)$ with parameters $p = \frac{\sqrt{2}}{1+\sqrt{2}}$, $q = 2$ and boundary conditions ξ.

We note that the exact location of the rectangle (having the coordinate axes as two sides) is not relevant, but only the size of the rectangle and the fact that the rectangle is parallel to the coordinate axes of the plane (we consider $\mathbb{L}^2$ to be also "parallel" to coordinate axes in the plane) are relevant.

7.4.1. *Planar duality with boundary conditions.* We define the (*square lattice*) *dual* of any subgraph G of the square lattice to be the dual square lattice subgraph G_d whose set of vertexes are the vertexes of the *dual square lattice* (that is, the infinite lattice formed by taking as vertexes the faces of the original square lattice and connecting two vertexes corresponding to two faces of the original lattice sharing a vertex) that are adjacent to an edge of the dual lattice being intersected by an edge of G; the set of edges of G_d are those edges of dual lattice intersected by the edges of G. Now we could try and formulate the dual model in G_d as we did in the case of general duality above in subsection Planar duality; this however would lead to problems, as the resulting dual model would not be a random cluster model. This is because it is not the case that the number of faces of the original graph is the same as the number of vertexes of the dual graph, a fact we used above in the general case (the other relevant fact used above, namely that the edge-sets of the original and dual graphs are in bijective correspondence, still holds). One can see this by for example considering G to be the border of a big square, id est, G would be the set of vertexes and edges encountered in a large, square-shaped lattice path. Then the number of faces of G in 2 whereas the number of vertexes of dual graph is not 2.

We will not study in general what conditions G must satisfy in order for the above defined dual model on the square lattice dual G_d to fulfil the demand that dual model is also a random cluster model as above. Instead we limit ourselves to study the case $G = \Lambda(n, m)$, the $n \times m$ straight rectangle, $n, m \in \mathbb{N}$.

So now we would like to define the dual model in $\Lambda(n, m)_d$ as above in subsection Planar duality. But we note that our problem still stands; the number of faces of the original graph is not the same as the number of vertexes of the dual. However we can make this work by invoking suitable boundary conditions of the random cluster model. Namely in the general duality case in the above subsection there was no boundary conditions, or equivalently, we only studied the case of free boundary conditions on the original and dual graphs. But if we now impose the free boundary conditions in the original and wired boundary conditions in the dual graph (the *boundary* of the dual graph being the vertexes that are in the infinite face of the original graph), the equality between the faces of the original configurations and the clusters of the dual configurations is again true. Therefore we say that the *dual boundary conditions* to free boundary conditions are the wired boundary conditions, and our dual model of the original random cluster model on the graph $\Lambda(n, m)$ with free boundary conditions is the

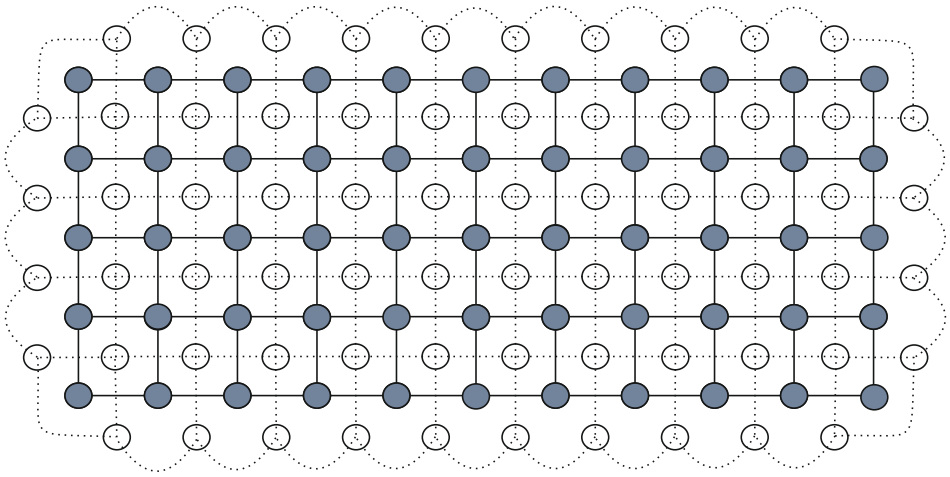


FIGURE 34. Here we see the rectangle $\Lambda(n,m)$ and its dual (in the random cluster model-sense) $\Lambda(n,m)_d$. In $\Lambda(n,m)$ we have free boundary conditions, and in $\Lambda(n,m)_d$ we have wired boundary conditions, represented by the curved edges.

random cluster model on the graph $\Lambda(n,m)_d$ with wired boundary conditions and the parameters p, p_d, q, q_d being related as in the general case, and dual configurations being defined as in the general case. Let us note that our above definition of the square lattice dual of a graph was done for the free boundary conditions in mind (that is, we actually assumed from the beginning that the boundary conditions in the original model are the free boundary conditions; we failed to explicitly say this), and actually in the subsection Planar duality above we considered that the boundary conditions of the original and dual random cluster models were free. If the boundary conditions on the original graph are something other than free, the construction of the dual graph must be altered so that the duality of the models can be achieved. One can show that the dual boundary condition to the wired boundary condition is the free boundary condition (for suitably chosen original and dual graphs). Also more general boundary conditions can be studied; we however do not attempt this here.

The duality of the random cluster models with boundary conditions taken into account also survives to the thermodynamic limit and can therefore be used in the infinite setting also; we however use only the "finite duality".

7.4.2. *Domain considerations: Dobrushin domains.* Let us define the *medial lattice* (or more appropriately, the *medial graph*) of a general planar graph G (there are many names used for this lattice in the literature, for example *surrounding lattice* [5] and *covering lattice* [112], [113]; let us note that some authors use the term "*covering lattice* (*graph*)" as a synonym for *line graph*, which is another graph, different of the medial graph, that can be constructed from a given graph). This lattice is formed by taking as vertexes

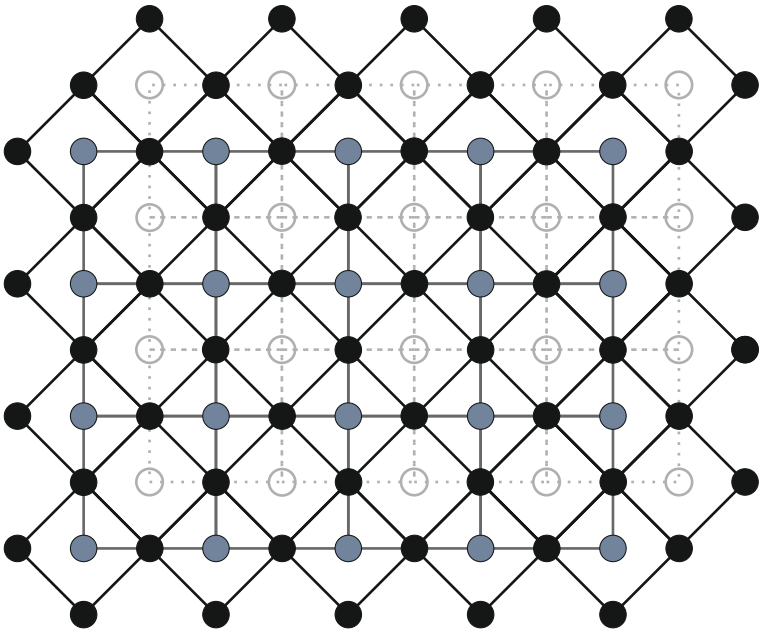


FIGURE 35. Part of the medial and dual lattices of the square lattice in two dimensions. The medial lattice is in black, the original square lattice is in grey, and the dual lattice is in light grey with dotted lines as edges.

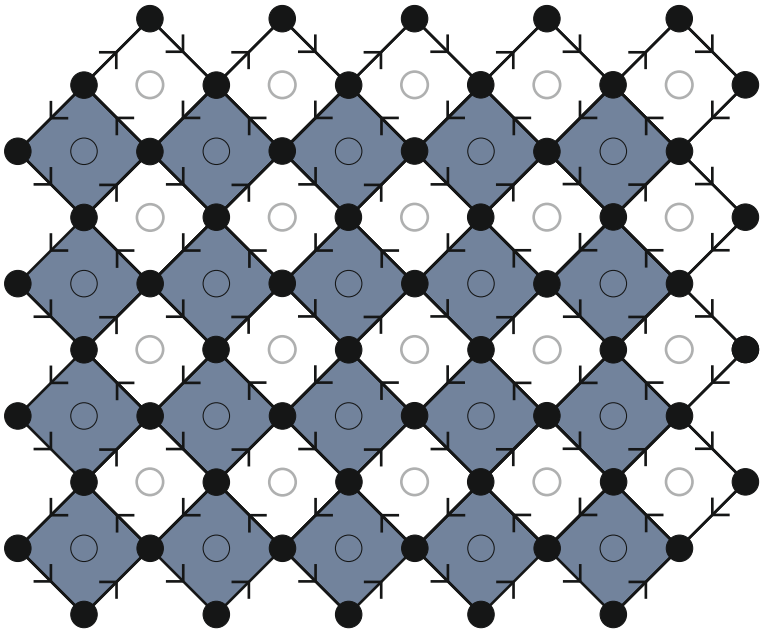


FIGURE 36. The orientation of the medial lattice in chessboard fashion.

the centres of edges of G, and by drawing the edges between these vertexes by going around each face (including the infinite face if there is any; in our case we have none) of the original graph and connecting with edges those vertexes that correspond to edges of the original graph that share an endpoint and are on the boundary of the face.

Note that this construction might yield multiple edges between medial vertexes in the general case (this happens when the edges of the original graph that correspond to the above mentioned vertexes of the medial graph have a common endpoint and two common faces on the original graph), but we can forget this since we are actually only interested in the case $G = \mathbb{L}^2$ where this does not happen, for we note that the medial lattice of $\mathbb{L}^2$ is isomorphic to the original $\mathbb{L}^2$ (subject to rotation by $\frac{\pi}{4}$ and scaling by $\frac{1}{\sqrt{2}}$). Now we shall color the medial lattice as follows: we see that the medial lattice consist of diamonds, and in the center of each diamond we have either a vertex of the original lattice, or the vertex of the dual lattice. We color the diamonds having in their center a vertex of the original lattice black and those that have a vertex of the dual lattice white, so that we cover the plane with chessboard colouring; see figure 35. Finally, we orientate the medial lattice such that the edges are oriented to circle the black diamonds in anticlockwise direction; see figure 36. We will refer to this chessboard colouring of the plane frequently, and we refer to vertexes of the original and dual lattices as black and white diamonds; given a primal (dual) lattice subgraph we for example say that two black (white) diamonds are *connected* (in the subgraph in question) if the corresponding vertexes are connected in the original (dual) lattice.

Now let us define *Dobrushin domains* in this setting (remember we already defined Dobrushin domains in the SLE-section above). Intuitively we want a Dobrushin domain to represent a finite, simply connected plane domain with two points a and b given from its boundary so that the boundary consists of two arcs $(a; b)$, the anti-clockwise arc of the boundary from a to b and $(b; a)$, the anti-clockwise arc of the boundary from b to a, respectively (similarly as in the SLE-section). Technically, let e_a and e_b be two edges of the medial lattice; let a and b be the corresponding sites of the original lattice (the sites whose black diamonds the edges border; obviously each medial lattice edge borders only one black diamond). Let ∂_{ab} and ∂_{ba} be two self-avoiding paths (meaning that the paths never use some edge twice; they can return twice to a vertex) of the medial lattice, both starting at e_a and ending at e_b and following the orientation of the medial lattice such that the paths intersect (paths intersect when they have a common edge or vertex) only at e_a and e_b and the vertexes of these edges; see figure 37. We also assume that the loop obtained by following ∂_{ab} in original and ∂_{ba} in reverse direction is oriented anti-clockwise. Finally we demand that the paths be such that the edge e_b points out of the above described loop (note that due to our definition of the paths ∂_{ab}, ∂_{ba} and the orientation of the loop they form, it must be the case that the edge e_a points inwards to the loop). We

make this demand because problems arise when the edge e_b is allowed to point inside the loop because this causes the free and wired boundaries (to be defined below) to intersect in the neighbourhood of e_b. Now the *medial lattice Dobrushin domain* $D_\diamond = (V_\diamond, E_\diamond)$ is the subgraph of the medial lattice contained in the above described loop including the paths ∂_{ab}, ∂_{ba}; see figure 38. The *boundary vertexes* $\partial V_\diamond$ of this domain is the set of vertexes belonging to paths ∂_{ab}, ∂_{ba}.

Now we define the *discrete Dobrushin domain* to be the subgraph $D = (V, E)$ of the primal lattice $\mathbb{L}^2$ such that V consists of sites whose black diamonds are bordered by some edge of $D_\diamond$, and E consists of edges between sites of V that do not intersect ∂_{ab} and of edges that connect two vertexes of the wired arc (to be defined below) that do intersect ∂_{ba} (meaning that when these edges meet the medial lattice path ∂_{ba} in a medial lattice vertex, the path enters and exits the vertex on the same side of the edge. Note that the medial lattice path may come back to the vertex and thus meet the edge twice; we apply the same condition of entering and exiting on the same side independently for both meetings); see figure 39. We define the *free arc* $(a; b)$ (respectively, the *wired arc* $(b; a)$) to be the the set of sites of $\mathbb{L}^2$ whose corresponding black diamonds are bordered by edges of ∂_{ab} (respectively, ∂_{ba}); see figures 37, 39. Now we define the dual D_d of domain D (again, not a graph-theoretic dual) to be a subgraph of the dual lattice that is formed from $D_\diamond$ completely analogously to D with black diamonds replaced by white diamonds and ∂_{ab} by ∂_{ba} in the definition; see figure 40. Now we call the set of white diamonds bordered by an edge of ∂_{ab} the *dual free arc* and the set of white diamonds bordered by an edge of ∂ba the *dual wired arc*; see figure 37. Note that the dual free arc lives on the dual lattice whereas the free arc lives on the primal lattice. Note also that because of our definition of the orientation of the medial lattice, all oriented medial paths have white diamonds to their right side and black diamonds on their left side, so that the wired arc and the dual free arc are "just outside" the domain $D_\diamond$; see figure 37. Now we define the medial lattice domain $D_{\partial\diamond}$ as the domain $D_\diamond$ with the diamonds of the wired arc and dual free arc added (that is, the edges and vertexes of the medial lattice corresponding to those diamonds added); see figure 41. Note that there is no problem with this definition as the wired and dual free arc are always separate arcs.

Now when given a Dobrushin domain D of the primal lattice with two marked vertexes of the "boundary" a and b such that its relation to $D_\diamond$ and D_d, also the choice of the paths ∂_{ab}, ∂_{ba} is imminent (these paths are the medial lattice paths that border D such that ∂_{ab} runs on the outer boundary of the part of the boundary of D from a to b in anti-clockwise direction and ∂_{ba} runs on the inner boundary of D from b to a in anti-clockwise direction), we denote this Dobrushin domain with boundary points a, b as (D, a, b).

Now we define a random cluster measure $P_{(D,a,b)}$ on the Dobrushin domain (D, a, b). We put wired boundary conditions on the wired arc, meaning that we connect all the vertexes pairwise, and free boundary conditions on

the free arc (in the case two vertexes of the primal lattice who share an edge are on both the free and wired arcs - this happens when the paths ∂_{ab} and ∂_{ba} follow the same trajectory at the borders of these vertexes - then we adopt the wired boundary condition between them); see figure 39. Note that the same kind of clustering as is produced by the above boundary conditions can be achieved by simply adding into the boundary conditions only those edges that join vertexes of the wired arc that are neighbouring vertexes in the primal lattice and that do not intersect ∂_{ba}; let us denote this "neighbours only" wired boundary conditions by ξ^N (we use this below for example when we form the loop representation of the random cluster model). Note also that according to our definition of the domain, the edges that connect vertexes of the wired arc in the primal lattice and that do not intersect ∂_{ba} are not taken into the domain whereas edges that connect vertexes of the free arc in the primal lattice are taken into the domain; this is why the free and wired boundary conditions on the respective arcs make sense. The boundary conditions above are called the *Dobrushin boundary conditions* on (D, a, b). The random cluster measure $P_{(D,a,b)}$ is the random cluster measure on D with these boundary conditions and parameters $p = \frac{\sqrt{2}}{1+\sqrt{2}}$, $q = 2$. Now for any random cluster configuration ω of D we define the corresponding configuration ω_d of D_d to be the configuration formed by making all the dual edges intersecting an open primal edge closed, and the others open. This definition differs slightly from the general dual model defined above in subsection Planar duality, but it has got the advantage that the dual model obeys random cluster probability measure with wired boundary conditions on the dual free arc and free boundary conditions on the dual wired arc, the *Dobrushin dual boundary conditions*; see figure 40. Again the similar kind of clustering in the dual model can be achieved by wiring together only those dual vertexes of the dual free arc that are neighbours in the dual lattice and whose connecting edge does not intersect ∂_{ab}; denote these boundary conditions ξ_d^N. Let us note in passing that our motivation for introducing the dual model is actually not the desire to study the dual model's probabilistic aspects; we use the dual model only as a geometrical auxiliary below.

Now for any configuration ω of the Dobrushin domain (D, a, b), the interfaces between the clusters of $\omega \cup \xi^N$ and $\omega_d \cup \xi_d^N$ (where we again mean that the boundary condition edges are to be added into the configuration) in the medial lattice form a family of loops and a path, called the *exploration path*, from e_a to e_b; see figure 42. The existence of the path is seen as follows: let A be the set of vertexes of D that can be connected to the vertexes of the wired boundary $(b; a)$ by an open path. Then the boundary of A, taken as a path in the medial lattice (note that we do not allow this path to go outside D, circling the boundary of D if needed), starts from e_a (because of the choice of e_a) and ends at e_b (because of the choice of e_b) and it is the wanted path.

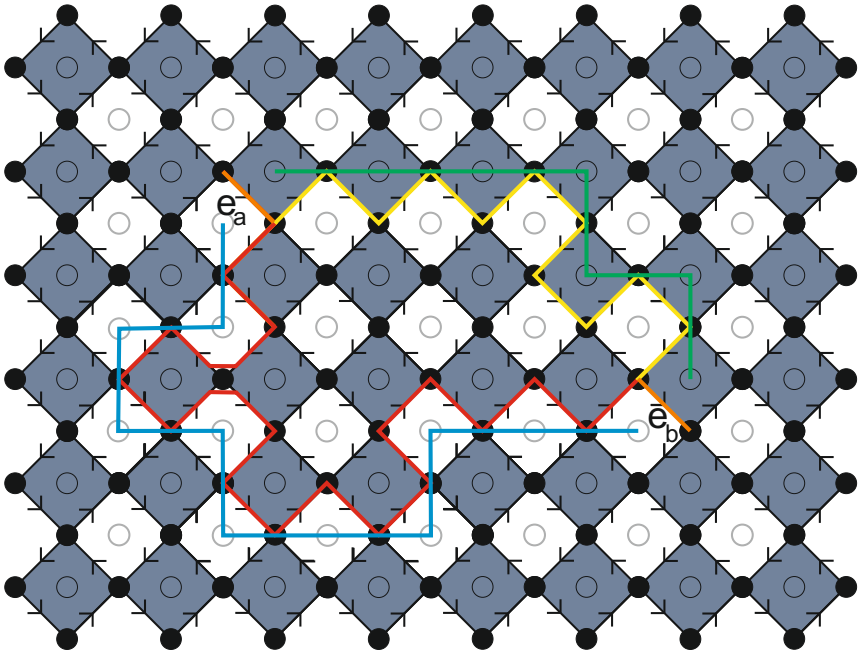


FIGURE 37. The situation in defining the medial lattice Dobrushin domain. The edges e_a, e_b are marked in the figure; the path ∂_{ab} is in red and the path ∂_{ba} is in yellow; when the paths return to the same vertex for the second time we denote the paths such that the paths appear non-self-intersecting; this is just a graphical aid. The blue and green paths denote the dual free and wired arcs, respectively.

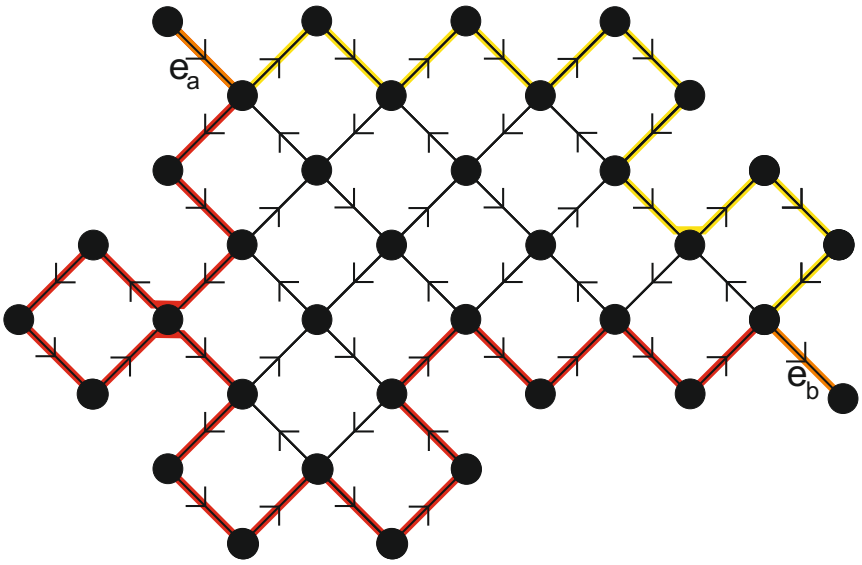


FIGURE 38. The medial lattice domain $D_\diamond$.

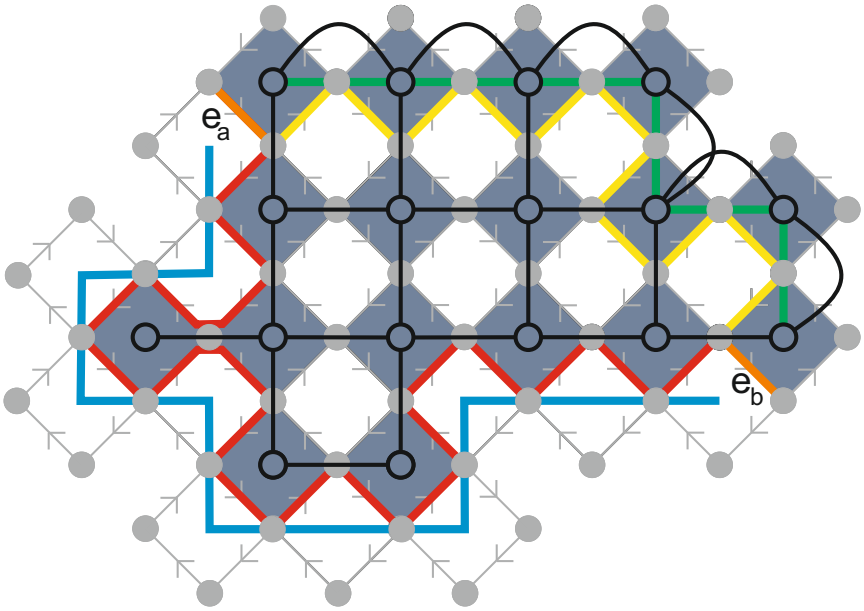


FIGURE 39. Here we have a Dobrushin domain of the primal lattice. The domain itself is in black, and the black arcs connecting vertexes on the "boundary" denote the boundary condition edges.

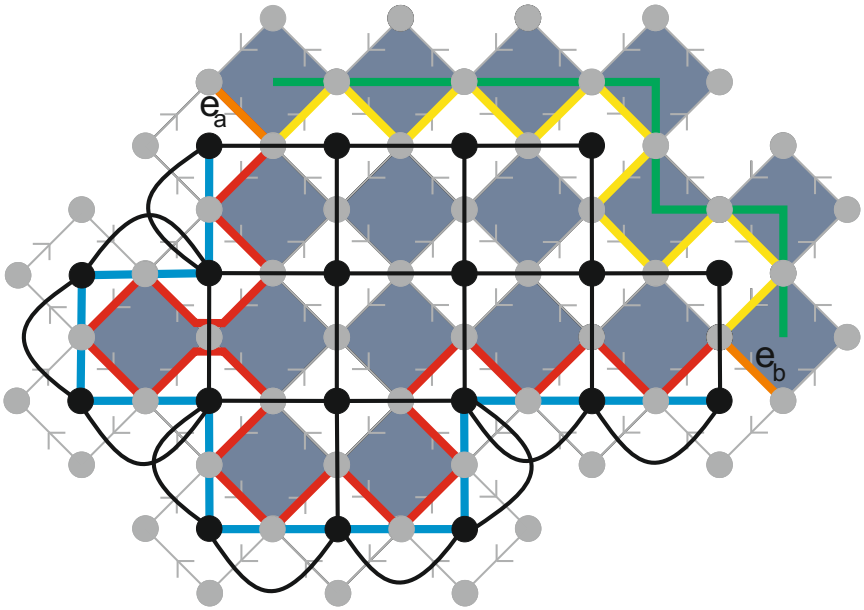


FIGURE 40. The dual lattice Dobrushin domain in black, with boundary condition edges denoted by black arcs.

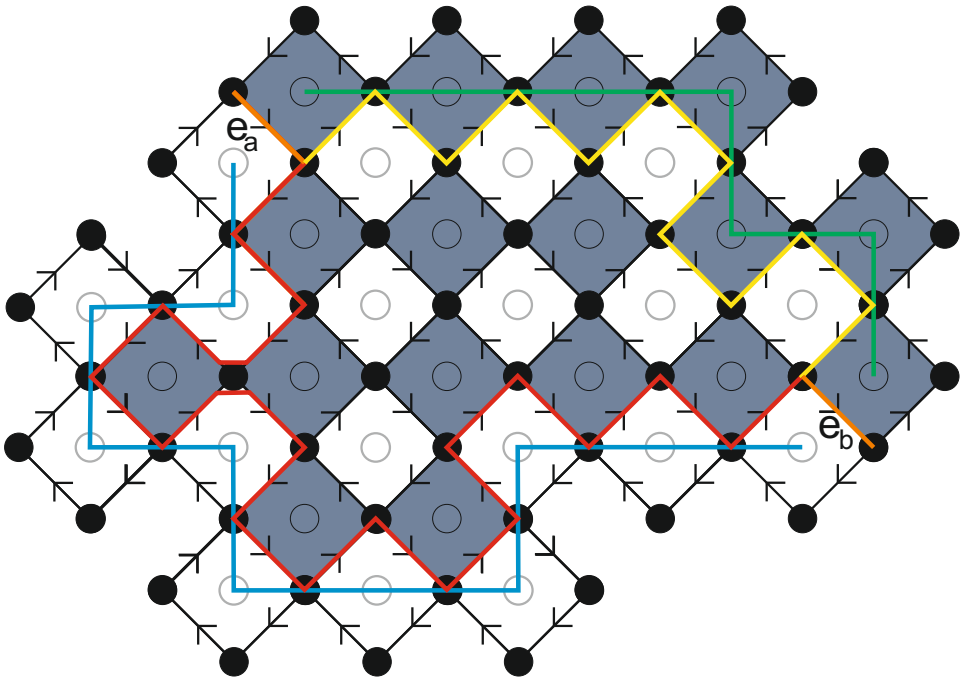


FIGURE 41. The medial lattice domain $D_{\partial\diamond}$.

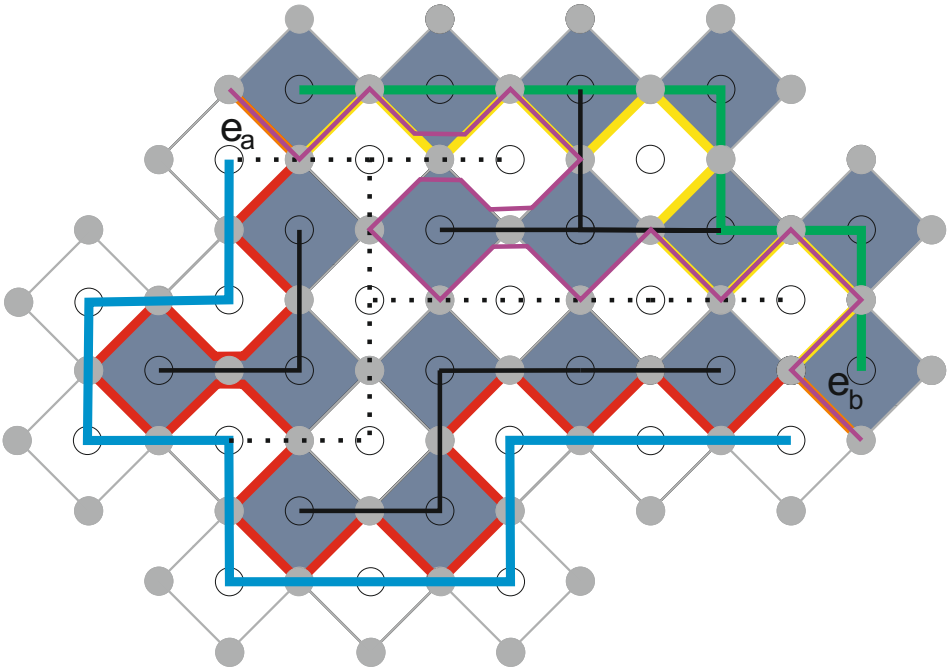


FIGURE 42. A random cluster model configuration on the primal lattice Dobrushin domain defines a configuration on the dual lattice Dobrushin domain and an exploration path (purple) on the medial lattice Dobrushin domain.

7.4.3. *Domain Markov property.* The random cluster model has a property called *Domain Markov property*; this property simply says that given a subdomain of the whole domain, the configuration in the subdomain is affected by the configuration outside the subdomain (this interaction is due to the cluster weight q in the random cluster measure) in such a way that the subdomain configuration follows a probability law that is the random cluster measure in the subdomain with boundary conditions that take into consideration the connections taking place outside the subdomain.

Namely, let $G = (V, E)$ be a finite graph and let $\emptyset \neq F \subset E$. Let $W \subset V$ be the set of endpoints of edges of F and let $G' = (W, F)$. Then for any boundary conditions ξ, the measure $P^{\xi}_{G,p,q}$ conditioned to match some configuration $\omega_{E\setminus F}$ on $E \setminus F$ is equal to $P^{\xi\cup\omega_{E\setminus F}}_{G',p,q}$ where $\xi \cup \omega_{E\setminus F}$ denotes the boundary conditions inherited from $\omega_{E\setminus F}$ and ξ; that is, two boundary vertexes of G' are connected in the boundary condition $\xi \cup \omega_{E\setminus F}$, if they are connected as vertexes in $\omega_{E\setminus F}$ with the additional edges of ξ (note that this means that the probability law is the same for configurations $\omega_{E\setminus F}$, $\omega'_{E\setminus F}$ that give G' a similar boundary). This can be seen by a direct calculation; let us use the notation $\omega_{E\setminus F}$ for both the configuration in $E \setminus F$ and for the set $\{\omega \in \Omega \,|\, \omega(e) = \omega_{E\setminus F}(e) \text{ for } e \in E \setminus F\}$ and let $\omega \in \Omega$ be such that $\omega \in \omega_{E\setminus F}$, that is, ω agrees with $\omega_{E\setminus F}$ in $E \setminus F$ (for configurations $\omega \in \Omega$, $\omega \notin \omega_{E\setminus F}$, the conditional probability below is just zero, as it should be). Then

$$
\begin{aligned}
P^{\xi}_{G,p,q}(\omega \,|\, \omega_{E\setminus F}) &= \frac{P^{\xi}_{G,p,q}(\{\omega\} \cap \omega_{E\setminus F})}{P^{\xi}_{G,p,q}(\omega_{E\setminus F})} \\
&= \frac{P^{\xi}_{G,p,q}(\omega)}{\sum_{\omega' \in \omega_{E\setminus F}} \frac{1}{Z^{\xi}_{G,p,q}} p^{|O(\omega')|}(1-p)^{|E|-|O(\omega')|} q^{C(\omega'\cup\xi)}} \\
&= \frac{\frac{1}{Z^{\xi}_{G,p,q}} p^{|O(\omega)|}(1-p)^{|E|-|O(\omega)|} q^{C(\omega\cup\xi)}}{\frac{1}{Z^{\xi}_{G,p,q}} \sum_{\omega' \in \omega_{E\setminus F}} p^{|O(\omega')|}(1-p)^{|E|-|O(\omega')|} q^{C(\omega'\cup\xi)}} \\
&= \frac{p^{|O(\omega_F)|+|O(\omega_{E\setminus F})|}(1-p)^{|F|+|E\setminus F|-\left(|O(\omega_F)|+|O(\omega_{E\setminus F})|\right)} q^{C(\omega_F\cup\omega_{E\setminus F}\cup\xi)}}{\sum_{\omega' \in \omega_{E\setminus F}} p^{|O(\omega'_F)|+|O(\omega_{E\setminus F})|}(1-p)^{|F|+|E\setminus F|-\left(|O(\omega'_F)|+|O(\omega_{E\setminus F})|\right)} q^{C(\omega'_F\cup\omega_{E\setminus F}\cup\xi)}} \\
&= \frac{p^{|O(\omega_{E\setminus F})|}(1-p)^{|E\setminus F|-|O(\omega_{E\setminus F})|} p^{|O(\omega_F)|}(1-p)^{|F|-|O(\omega_F)|} q^{C(\omega_F\cup\omega_{E\setminus F}\cup\xi)}}{p^{|O(\omega_{E\setminus F})|}(1-p)^{|E\setminus F|-|O(\omega_{E\setminus F})|} \sum_{\omega' \in \omega_{E\setminus F}} p^{|O(\omega'_F)|}(1-p)^{|F|-|O(\omega'_F)|} q^{C(\omega'_F\cup\omega_{E\setminus F}\cup\xi)}} \\
&= \frac{p^{|O(\omega_F)|}(1-p)^{|F|-|O(\omega_F)|} q^{C(\omega_F\cup(\omega_{E\setminus F}\cup\xi))}}{\sum_{\omega' \in \omega_{E\setminus F}} p^{|O(\omega'_F)|}(1-p)^{|F|-|O(\omega'_F)|} q^{C(\omega'_F\cup(\omega_{E\setminus F}\cup\xi))}} \\
&= \frac{p^{|O(\omega_F)|}(1-p)^{|F|-|O(\omega_F)|} q^{C(\omega_F\cup(\omega_{E\setminus F}\cup\xi))}}{Z^{\xi\cup\omega_{E\setminus F}}_{G',p,q}} \\
&= P^{\xi\cup\omega_{E\setminus F}}_{G',p,q}(\omega_F)
\end{aligned}
$$

where ω_F is the configuration in G'.

Now let us make a remark: if we have a Dobrushin domain (D, a, b), then the *slit domain* created by "removing" some $T \in \mathbb{N}$ first steps of the exploration path is also a Dobrushin domain. In more detail: given a Dobrushin domain and a configuration in it giving us an exploration path, let l be the set of vertexes of D adjacent to the initial segment of the exploration path γ, that is, adjacent to path $\gamma([0, T] \cap \mathbb{N})$, where $T \in \mathbb{N}$ and the exploration path γ of the medial lattice is parametrized with the number of steps it takes (each time the exploration path goes from one medial edge to another is counted as a step); see figure 43. A vertex of the original lattice is adjacent to a path of the medial lattice if the path travels through an edge that is in the boundary of the diamond corresponding to the vertex. We abuse the notation and denote with $\gamma(T)$ the vertex of the primal lattice whose diamond the last medial edge of the path $\gamma([0, T]\cap\mathbb{N})$ borders, and we also denote the exploration path $\gamma(A \cap \mathbb{N}) = \gamma A$ for all intervals (open, closed or otherwise) $A \subset \mathbb{R}$. Let us now define a new domain $(D \setminus \gamma[0, T[, \gamma(T), b)$ by setting new Dobrushin boundary arcs $(\gamma(T); b)$, $(b; \gamma(T))$, meaning that given paths ∂_{ab}, ∂_{ba} of the original Dobrushin domain (D, a, b), we extend the paths so that they begin at the T:th ($T \in \mathbb{N}$) edge of the exploration path, and "bounce along" the exploration path until they hit an edge that was a part of them before the extension, and after that they continue to follow their old trajectory; see figures 44, 45. These new Dobrushin boundary arcs, denote them $\partial_{\gamma(T)b}$, $\partial_{b\gamma(T)}$ may be such that they have a common edge (but because of the topology of the planar simply connected Dobrushin domain (D, a, b), these paths cannot have a common vertex without having a common edge; this is basically because if two medial lattice path have a common vertex but not a common edge, then these two paths must be travelling in different direction at this vertex; but if these paths are such that they connect the same two points x and y, or in our case these are the edges e_a and e_b, such that both path travel in the direction from x to y, say, and they do not have a common edge and they do not travel any edge twice, the this "travelling at different directions at the same point" leads to a contradiction, since these kind of planar path do not exist - one of the paths must intersect either itself or the other path to accomplish this) at other places than just at the edges $\gamma(T)$ and e_b; if this is the case, we redefine these boundary arcs such that they only consist of those final segments beginning at the edge that is the second-to-last common edge to both paths (e_b is obviously the last common edge of both paths); we note that this redefinition is justified by the Domain Markov property, or to be more precise, the "domain slicing property" that we shall introduce below, since we see that when the Dobrushin boundary arcs intersect, then when we put the Dobrushin boundary conditions along the arcs, the boundary conditions are such that they "slice" the domain into independent domains (each separate loop between two intersections of the Dobrushin boundary arcs becoming an independent domain) which do not interact in the random cluster measure via the cluster weight (of course, the only interactions of separated areas in the random cluster model must

take place through the cluster weight q). Having defined the new Dobrushin boundary arcs, we continue the definition of the new Dobrushin domain as above when we defined the Dobrushin domains in general. By "bouncing along" the exploration path we actually mean that $\partial_{b\gamma(T)}$ follows the exploration path in reverse on the left side of the exploration path (the directions are given as seen from the exploration path looking in the direction of the exploration path) starting from the T:th edge of the exploration path, and $\partial_{\gamma(T)b}$ follows the exploration path in reverse on the right side, so that the paths $\partial_{\gamma(T)b}$, $\partial_{b\gamma(T)}$ have no vertexes in common (apart of the vertexes adjacent to the medial edge $\gamma(T)$) and no edge of the exploration path is in either of paths $\partial_{\gamma(T)b}$, $\partial_{b\gamma(T)}$, and there are no more than three edges on either paths between two common vertexes of the exploration path and the new (compared to ∂_{ab}, ∂_{ba}) extension segment of the paths $\partial_{\gamma(T)b}$, $\partial_{b\gamma(T)}$; see figure 44. This means that the paths $\partial_{\gamma(T)b}$ and $\partial_{b\gamma(T)}$ follow the "other sides" (other than the exploration path) of some of the diamonds (black in the case of $\partial_{b\gamma(T)}$ and white in the case of $\partial_{\gamma(T)b}$) that belong to the new domain (as some of the original domain might have been cut away) whose edges belong to the exploration paths initial T-segment. The diamonds whose edges are joined to these extensions of ∂_{ab} and ∂_{ba} are those that form the "right" and "left boundary" of the exploration path, id est, those diamonds who have a medial lattice edge on their boundary that is on the exploration path and who also have an edge on their boundary that is not on the exploration path (the path $\partial_{b\gamma(T)}$ consist of boundary edges of such black diamonds and the path $\partial_{\gamma(T)b}$ consists of boundary edges of such white diamonds); see figure 44. Note that this definition leads to the wired arc of the new domain consisting of possibly some (not all) vertexes of l that are in the new domain (and of some of the vertexes of the wired arc of the old domain) and the dual free arc consisting of possibly some (not all) of the dual vertexes that are part of the dual of the new domain whose white diamonds are next to (have a common edge with) the black diamonds of the vertexes of l (and of some of the vertexes of the dual free arc of the old domain). Note also that in the case the exploration path comes to the boundary (id est, meets ∂_{ab} or ∂_{ba}) and the stopping is made (the $T \in \mathbb{N}$ is chosen) such that $\gamma(T)$ is an edge of the path ∂_{ab} (or ∂_{ba}), then the new extension of the path ∂_{ab} (∂_{ba}) just follows the segment of the old path starting from $\gamma(T)$. Now, conditionally on γ, the FK Ising law in the new domain is exactly $P_{D\setminus\gamma[0,T[,\gamma(T),b}$.

Let us also introduce a "*domain slicing property*" (term coined by the author; this is actually just a special case of the Domain Markov property). Namely, when we impose such boundary conditions (wired or dual wired, see below) on our random cluster model that they "slice" the domain in two, then these two parts of the domain become independent in the random cluster measure, meaning that events taking place in one slice do not depend on the events taking place in the other.

Let us demonstrate this property in a special case that will be useful below. Namely let us study a rectangular domain $G = (V, E)$ whose sides are

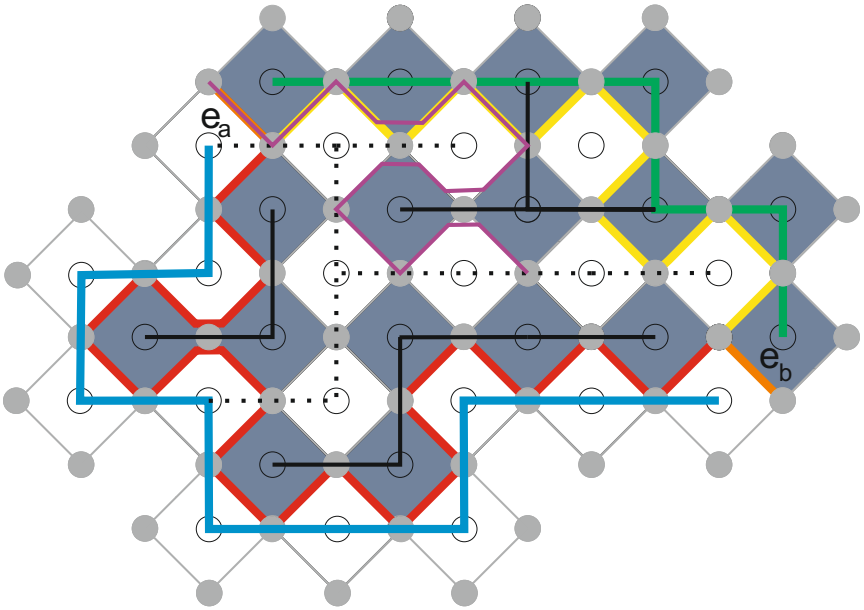


FIGURE 43. The domain Markov property of the random cluster model. In this first step, we take some initial segment of the exploration path.

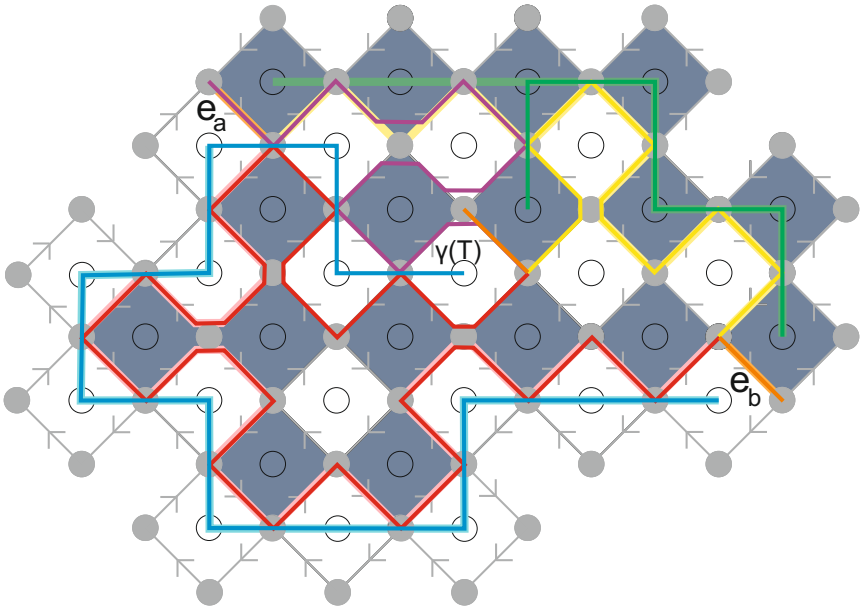


FIGURE 44. The domain Markov property of the random cluster model. In the second step, we form the medial lattice domain corresponding to $(D \setminus \gamma[0,T[, \gamma(T), b)$.

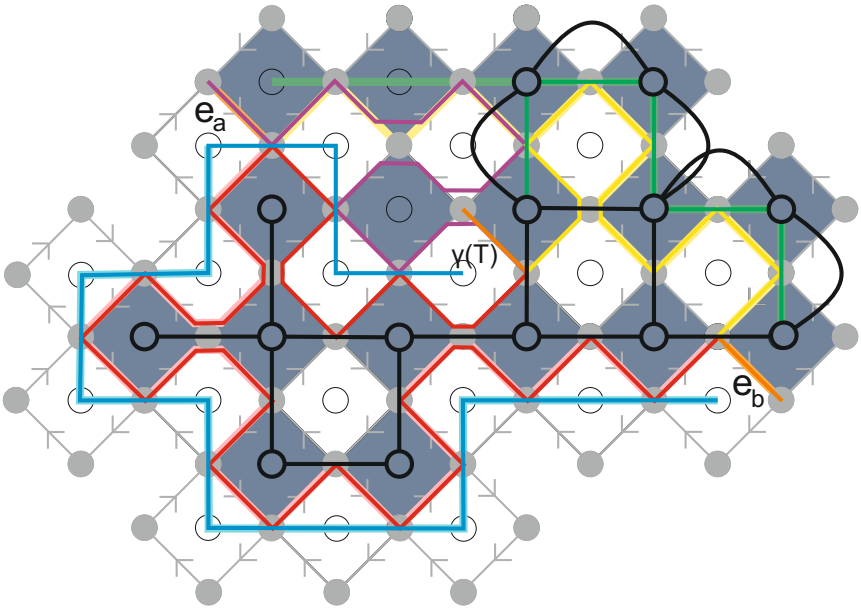


FIGURE 45. The domain Markov property of the random cluster model. In the last step, we form the primal lattice domain $(D \setminus \gamma[0,T[, \gamma(T), b)$.

parallel to the coordinate axes and let us impose boundary conditions ξ such that all the vertexes of the rectangle along a vertical line that is inside the rectangle (vertexes having the same x-coordinate such that this coordinate value is not maximal or minimal) are wired together; see figure 46. Now let A be the part of G left or right from this wired vertical line (the edges of the wired line can be counted into A or left out). We shall show that the configurations ω_A in A are distributed according to random cluster measure in A with appropriate boundary conditions; let us begin by manipulating the normalizing constant:

$$\begin{aligned}
Z^{\xi}_{G,p,q} &= \sum_{\omega\in\Omega_G} p^{|O(\omega)|}(1-p)^{|E|-|O(\omega)|}\, q^{C(\omega\cup\xi)} \\
&= \sum_{\omega\restriction A\in\Omega_A} \sum_{\omega\restriction G\setminus A\in\Omega_{G\setminus A}} p^{|O(\omega\restriction A)|+|O(\omega\restriction G\setminus A)|}(1-p)^{|E_A|+|E_{G\setminus A}|-|O(\omega\restriction A)|-|O(\omega\restriction G\setminus A)|} \\
&\quad \cdot q^{C(\omega\restriction A\cup\xi_A)+C(\omega\restriction G\setminus A\cup\xi_{G\setminus A})-1} \\
&= \sum_{\omega\restriction A\in\Omega_A} p^{|O(\omega\restriction A)|}(1-p)^{|E_A|-|O(\omega\restriction A|}\, q^{C(\omega\restriction A\cup\xi_A)}\cdot q^{-1} \\
&\quad \cdot \sum_{\omega\restriction G\setminus A\in\Omega_{G\setminus A}} p^{|O(\omega\restriction G\setminus A)|}(1-p)^{|E_{G\setminus A}|-|O(\omega\restriction G\setminus A)|}\, q^{C(\omega\restriction G\setminus A\cup\xi_{G\setminus A})} \\
&= Z^{\xi_A}_{A,p,q}\cdot q^{-1}\cdot Z^{\xi_{G\setminus A}}_{G\setminus A,p,q}.
\end{aligned}$$

So we get for the probability measure:

$$\begin{aligned}
P^{\xi}_{G,p,q}(\omega_A) &= \sum_{\omega\in\omega_A} \frac{1}{Z^{\xi}_{G,p,q}} p^{|O(\omega)|}(1-p)^{|E|-|O(\omega)|} q^{C(\omega\cup\xi)} \\
&= \frac{1}{Z^{\xi}_{G,p,q}} \sum_{\omega\in\omega_A} p^{|O(\omega_A)|+|O(\omega\restriction G\setminus A)|}(1-p)^{|E_A|+|E_{G\setminus A}|-|O(\omega_A)|-|O(\omega\restriction G\setminus A)|} \\
&\quad \cdot q^{C(\omega_A\cup\xi_A)+C(\omega\restriction G\setminus A\cup\xi_{G\setminus A})-1} \\
&= \frac{1}{Z^{\xi}_{G,p,q}} p^{|O(\omega_A)|}(1-p)^{|E_A|-|O(\omega_A)|} q^{C(\omega_A\cup\xi_A)-1} \\
&\quad \cdot \sum_{\omega\restriction G\setminus A\in\omega_{G\setminus A}} p^{|O(\omega\restriction G\setminus A)|}(1-p)^{|E_{G\setminus A}|-|O(\omega\restriction G\setminus A)|} q^{C(\omega\restriction G\setminus A\cup\xi_{G\setminus A})} \\
&= \frac{1}{Z^{\xi_A}_{A,p,q}\cdot q^{-1}\cdot Z^{\xi_{G\setminus A}}_{G\setminus A,p,q}} p^{|O(\omega_A)|}(1-p)^{|E_A|-|O(\omega_A)|} q^{C(\omega_A\cup\xi_A)-1}\cdot Z^{\xi_{G\setminus A}}_{G\setminus A,p,q} \\
&= \frac{1}{Z^{\xi_A}_{A,p,q}} p^{|O(\omega_A)|}(1-p)^{|E_A|-|O(\omega_A)|} q^{C(\omega_A\cup\xi_A)} \\
&= P^{\xi_A}_{A,p,q}(\omega_A),
\end{aligned}$$

where we denote by ω_A a configuration of A and the set of those configurations of G agreeing with this configuration at A, and $\omega \restriction A$, $\omega \restriction G\setminus A$ are the restrictions of the configuration ω of G to A, $G\setminus A$, where $G\setminus A$ is simply the graph of those edges of G not in A and their endpoints (the wired line can counted into A or excluded from A); ξ_A, $\xi_{G\setminus A}$ are those edges of ξ connecting vertexes of A or $G\setminus A$ and we have noted

$$C(\omega\cup\xi) = C(\omega\restriction A\cup\xi_A) + C(\omega\restriction G\setminus A\cup\xi_{G\setminus A}) - 1,$$

because the component of the vertical wired line is counted twice on the right hand side. Note that the only relevant part of the above calculation is noting that the cluster weight divides into two components and that this is true simply because the boundary conditions divide the domain such that all clusters are either in A or in $G\setminus A$ except the component of the vertical wired line, and no other geometrical information about the boundary conditions apart from this division-property was needed; therefore it is easy to see how a more general case (more general boundary conditions on a non-rectangular domain) can be handled similarly. Note also that the restriction of the configurations in the set ω_A to $G\setminus A$ contain every configuration of $G\setminus A$.

Similarly, we could instead of a wired vertical line impose a dual wired vertical line, id est, we could demand that in the dual domain of the rectangle G there is a dual open vertical line of the dual lattice slicing G in two (A is the left or right part of G such that the edges crossing the dual open line are not included); see figure 46. This demand of course means precisely that we demand that all the horizontal edges of G that intersect this vertical line of the dual lattice are closed. Now we claim the probability of events in

A conditioned on the existence of dual open line (event D, that is, the event that all the edges crossing the dual open line are closed) follow the random cluster measure on A with the original boundary conditions restricted to A. Since

$$P^{\xi}_{G,p,q}(\omega_A \mid D) = \frac{P^{\xi}_{G,p,q}(\omega_A \cap D)}{P^{\xi}_{G,p,q}(D)},$$

let us study the enumerator and the denominator of the above formula separately. First the enumerator:

$$\begin{aligned}
P^{\xi}_{G,p,q}(\omega_A \cap D) &= \sum_{\omega \in \omega_A \cap D} \frac{1}{Z^{\xi}_{G,p,q}} p^{|O(\omega)|} (1-p)^{|E|-|O(\omega)|} q^{C(\omega \cup \xi)} \\
&= \sum_{\omega \in \omega_A \cap D} p^{|O(\omega_A)|+|O(\omega \restriction G\setminus A)|} (1-p)^{|E_A|+|E_{G\setminus A}|-|O(\omega_A)|-|O(\omega \restriction G\setminus A)|} \\
&\quad \cdot q^{C(\omega_A \cup \xi_A)+C(\omega \restriction G\setminus A \cup \xi_{G\setminus A})} \\
&= \left(p^{|O(\omega_A)|} (1-p)^{|E_A|-|O(\omega_A)|} q^{C(\omega_A \cup \xi_A)} \right) \\
&\quad \cdot \left(\sum_{\omega \restriction G\setminus A \in \Omega_{G\setminus A} \cap D} p^{|O(\omega \restriction G\setminus A)|} (1-p)^{|E_{G\setminus A}|-|O(\omega \restriction G\setminus A)|} q^{C(\omega \restriction G\setminus A \cup \xi_{G\setminus A})} \right)
\end{aligned}$$

and then the denominator:

$$\begin{aligned}
P^{\xi}_{G,p,q}(D) &= \sum_{\omega \in D} \frac{1}{Z^{\xi}_{G,p,q}} p^{|O(\omega)|} (1-p)^{|E|-|O(\omega)|} q^{C(\omega \cup \xi)} \\
&= \sum_{\omega \restriction A \in \Omega_A} \sum_{\omega \restriction G\setminus A \in \Omega_{G\setminus A} \cap D} p^{|O(\omega \restriction A)|+|O(\omega \restriction G\setminus A)|} (1-p)^{|E_A|+|E_{G\setminus A}|-|O(\omega \restriction A)|-|O(\omega \restriction G\setminus A)|} \\
&\quad \cdot q^{C(\omega \restriction A \cup \xi_A)+C(\omega \restriction G\setminus A)} \\
&= \left(\sum_{\omega \restriction A \in \Omega_A} p^{|O(\omega \restriction A)|} (1-p)^{|E_A|-|O(\omega \restriction A)|} q^{C(\omega \restriction A \cup \xi_A)} \right) \\
&\quad \cdot \left(\sum_{\omega \restriction G\setminus A \in \Omega_{G\setminus A} \cap D} p^{|O(\omega \restriction G\setminus A)|} (1-p)^{|E_{G\setminus A}|-|O(\omega \restriction G\setminus A)|} q^{C(\omega \restriction G\setminus A)} \right)
\end{aligned}$$

Dividing the enumerator by the denominator gives:

$$\frac{P^{\xi}_{G,p,q}(\omega_A \cap D)}{P^{\xi}_{G,p,q}(D)} = \frac{p^{|O(\omega_A)|} (1-p)^{|E_A|-|O(\omega_A)|} q^{C(\omega_A \cup \xi_A)}}{\sum_{\omega \restriction A \in \Omega_A} p^{|O(\omega \restriction A)|} (1-p)^{|E_A|-|O(\omega \restriction A)|} q^{C(\omega \restriction A \cup \xi_A)}} = P^{\xi_A}_{A,p,q}(\omega_A)$$

where denotation is similar as above. Note that again, no other geometrical information about the conditioning was used except the fact that the conditioning imposes factorization in the probability measure; therefore this argument can easily be extended into more general cases of boundary conditions and underlying domains.

Final note: the wired boundary conditions are actual boundary conditions on the graph G, whereas the dual wired boundary conditions mean that we

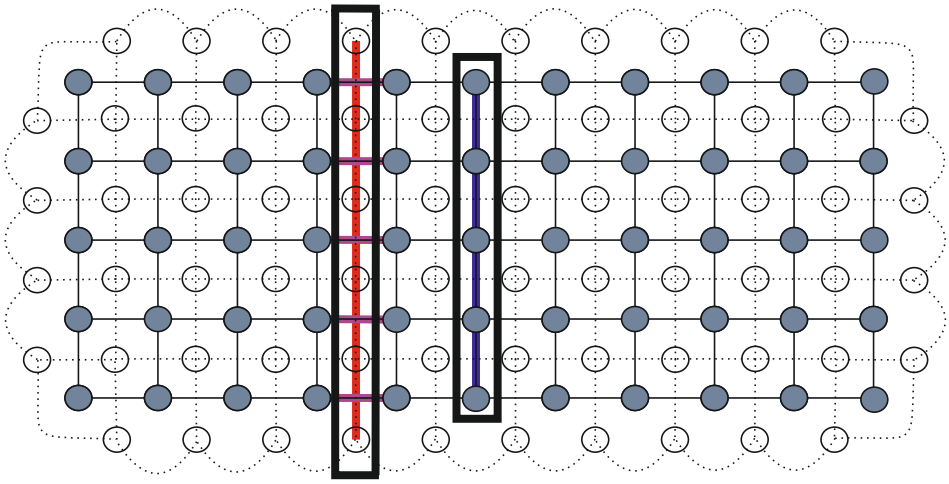


FIGURE 46. The domain slicing with a blue wired vertical line and a red dual wired vertical line (note the purple edges crossing the dual wired line; we actually demand these edges are closed).

actually condition everything on the event that the edges of the primal lattice crossing the dual wired boundary condition's dual edges are closed.

7.4.4. *Fermionic observable.* The (*Smirnov*) *fermionic observable* was discovered by *Stanislav Smirnov* who used it in his studies of percolation and random cluster model. The results that we are about to present are mostly due to Smirnov and he has presented them in his article [72]. Many of the following results arise from the property of *discrete holomorphicity* of the observable (Smirnov prefers the term *preholomorphicity*). Discrete holomorphicity and discrete complex analysis in general have found many important applications in the study of 2-dimensional probabilistic lattice models, especially when the scaling limits of various objects of interest in lattice models are studied, but since we do not need to develop this theory any further to accomplish our objectives in proving the box-crossing property, we do not delve into the subject any further.

Let (D, a, b) be a Dobrushin domain and let γ_ω be the exploration path in the configuration $\omega \in \Omega$. Let us define the *winding* $W(\Gamma, z, z')$ of the medial lattice path Γ between two medial lattice edges z,z' of the path as the total rotation (with sign) in radians the path makes from center of the edge z to the center of the edge z' (that is, we count in the edges z and z'). Whenever the path arrives at a vertex in the medial lattice, it has two directions where it can continue (due to the orientation of the medial lattice): left or right (the path cannot continue straight onwards, and due to the orientation of the medial lattice, directions "left" and "right" make sense, since they are observed from the direction of entry to the vertex), with the left-direction adding a $\frac{\pi}{2}$ to the winding and the right-direction adding a $-\frac{\pi}{2}$ to the winding.

We define now the (*Smirnov*) *fermionic observable* (*observable* being just a name for something whose value can be evaluated by just looking at the configuration $\omega \in \Omega$) F by formula:

$$F(e) = E_{(D,a,b)}(e^{-\frac{1}{2}iW(\gamma_\omega, e_a, e)}\chi_{e\in\gamma}) \tag{7.14}$$

for any edge e on the medial lattice that is in the domain $D_\diamond$ (given a Dobrushin domain (D, a, b)). Note that the observable is a deterministic, complex valued function of the edges of the medial lattice. The constant $\frac{1}{2}$ is called *spin*. The value $\frac{1}{2}$ corresponds to the value $q = 2$, and the overall form of the observable is chosen with square lattice at criticality in mind (for generalizations see [72]). The fermionic observable has the following property:

Lemma 7.2. *For any vertex $v \in V_\diamond$ in the medial lattice domain $D_\diamond$ with four adjacent edges, we have*

$$F(e_1) + F(e_3) = F(e_2) + F(e_4) \tag{7.15}$$

where the edges e_1, e_2, e_3, e_4 are the edges adjacent to v indexed in anticlockwise order starting from the upper right.

Remark: *Vincent Beffara* calls the above (7.15) the "*flow relation*" or "*flow condition*", see [6]. We shall use this term also.

In order to prove the above lemma, we need to take a little detour and present the *loop* (*gas*) *representation of random cluster model*. This means that we relate the random cluster model to a *loop* (*gas*) *model* or *polymer model* (there are many *loop models*, see [23], [64]), that is, a model in which the configurations represent some loops or paths (exactly what kind of loops or paths depends on the specific model) in a given lattice and the probability of a configuration ω is given by

$$\frac{1}{Z}w(\omega)t^{|\{\text{edges of the lattice covered by the loops}\}|}$$

where $w(\omega)$ is the *weight* given to the configuration; the weight depends on the specific model, and the weight can depend on, for example, the number of loops, the number of ways to color the planar "picture" given by the loop configuration (such that the distinct areas of the plane, separated by the loop borders, are coloured with one color each such that no two areas of same color are next to each other). The parameter t is the "*edge-weight*" (not all models have edge-weight; then one can just choose $t = 1$) and Z is the normalizing constant.

The loop model with which we shall couple the random cluster model is sometimes called *O(n) loop model*, the reason for the name being that the O(n) model discussed above can be reformulated as a similar loop model [48] (the loop formulation of O(n) model is only slightly more general loop model than this model); in physics literature this model and sometimes other loop models also are known with old-fashioned terms "*ring polymers*" or "*closed self-avoiding random walks*" and it might be called a *fully packed*

loop (*FPL*) *model*, since it turns out that every vertex of the lattice is visited by at least one loop (this is the property that the general O(n) model reformulation does not have) so that the loop configuration is Hamiltonian. As a remark we want to state that the usual feature of interest in loop models is the *critical point*, id est, the parameter value (or values) at which the model experiences phase transition from a dilute phase to a dense phase; in *dilute phase*, only a measure zero set of sites in the lattice are on average touched by a loop, and in *dense phase*, only a measure zero set of sites in the lattice are on average not touched by a loop.

In *O(n) loop model* the loops do not touch (that is, share an edge) and they do not intersect one another. The weight $w(\omega)$ of the O(n)-model is taken to be $n^{|\{\text{loops}\}|}$. Also we invoke a requirement that every edge of the graph must be part of exactly one loop; hence our model is called *completely packed loop* (*CPL*) *model*, id est, we consider Eulerian loop configurations. The loops we consider are called *polygons* in some texts (especially those in the physical literature; the term is also used in knot theoretic context).

So, our loop model is as follows: we consider loop configurations defined on the medial lattice, or to be more precise, on the graph $D_\diamond$, that are Eulerian loop configurations that contain one path such that this path connects e_a to e_b. These loops shall not intersect or touch each other apart from having a common vertex (that is, no edge can belong to two loops). It is clear that since $D_\diamond$ is finite, there is only a finite number of described loop configurations. The coupling is as follows: to any random cluster configuration ω we pair a loop configuration σ (we use the symbol σ just to avoid confusion) defined as follows: the configuration $\omega \cup \xi^N$ defines a set of clusters on the original and on the dual lattice (through the configuration $\omega_d \cup \xi_d^N$). Through each vertex of the medial lattice (the medial lattice vertexes are located on the intersection of the edges of the original and dual lattice, or on the edges connecting wired arc vertexes or dual edges connecting dual free arc vertexes) goes either an open edge of the original lattice or a dual open edge of the dual lattice, where the boundary condition edges are understood as open and the dual boundary condition edges as dual open. Now we draw the Eulerian loops on the medial lattice as interfaces separating the primal and dual clusters, that is, at each vertex of the medial lattice, the loop makes a $\frac{\pi}{2}$ turn such that it does not cross an open primal or dual edge (we form the loops by starting with any edge and moving forward in the directed lattice following the above rule; eventually we form a loop or in case we started from an edge that is to become a part of the exploration path, we end up at e_b and form a path. Then we pick another edge that is not yet part of a path or loop and continue construction, such that if we hit the final segment of the exploration path at some time, we extend the exploration path by adding the piece of path with which we are currently performing this process of making the loop configuration to the exploration path. Eventually each edge is part of a path or loop). This leads to the formation of loops surrounding the primal and dual clusters and the formation of an interface from e_a to e_b. The

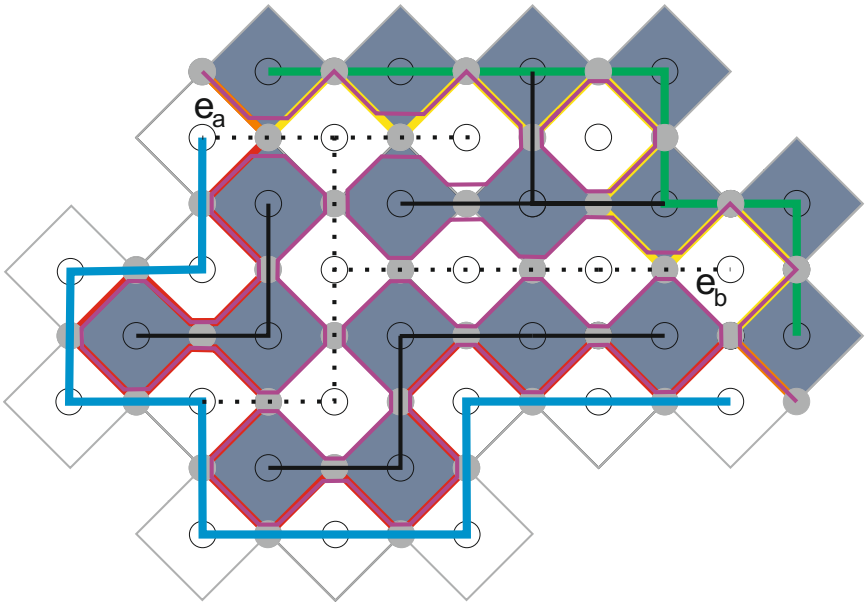


FIGURE 47. The random cluster configuration of the primal lattice Dobrushin domain (D, a, b) defines a loop configuration on the medial lattice Dobrushin domain $D_\diamond$.

interface is formed from e_a to e_b because one cannot form a loop around the component of the wired arc vertexes of the original lattice (a and b belong to wired arc), but what is formed instead is "half" of that loop, that is, a path goes around the component on the domain's side; this "half-loop" ends at e_a and e_b, since those edges "limit" the loop around the component (the loop cannot extend any further than e_a and e_b, and due to the way the component containing a and b are situated on the border of the Dobrushin domain, it is imminent that e_a and e_b are a part of the "attempted" loop around this component). Note that the interface is the interface between the component of the wired arc and the dual component of the dual free arc (the above argument can also be done on the dual lattice with the dual free arc component being the ones that are "attempted" to loop around). This gives our bijection between the configurations ω and σ; given any ω, we can now form a σ, and given a configuration σ, we can form the configuration ω by thinking that σ encodes the interfaces in ω; note that it is no problem to us that the set of interfaces usually admits two configurations (if a configuration has a given set of interfaces, the configuration that is formed from the previous one by changing the primal clusters to dual clusters and vice versa has the same set of interfaces) since we know the location of primal and dual vertexes with respect to the medial lattice, and hence we cannot confuse the primal and dual clusters with each other. Note also that since our medial lattice is oriented, we need not worry about the different possibilities of orientating the loops. This bijection is the loop representation of the random cluster model.

Now the probability of configuration ω can be expressed in terms of loops as follows: the probability of the configuration ω of the random cluster model is

$$P_{(D,a,b)}(\omega) = \frac{1}{Z} q^{C(\omega \cup \xi^N)} \prod_{e \in E} p^{\omega(e)} (1-p)^{1-\omega(e)},$$

where again $Z = Z(D,a,b) = \sum_{\omega \in \Omega} q^{C(\omega \cup \xi^N)} \prod_{e \in E} p^{\omega(e)} (1-p)^{1-\omega(e)}$ is the normalizing constant. Now the Euler's formula tells us that in the graph defined by the open edges of the configuration $\omega \cup \xi^N$ (a general finite planar graph; note that the reason we excluded the edges of ξ^n from the primal lattice Dobrushin domain (D,a,b) was actually that we wanted to make sure that the configurations $\omega \cup \xi^N$ are planar and this is obviously not the case if you include some of the edges of ξ^N, that is, some of those edges of the primal lattice connecting two vertexes of the wired arc such that the edge does not intersect ∂_{ba}, in (D,a,b)) we have

$$|V| - |E(\omega \cup \xi^N)| + F(\omega \cup \xi^N) - C(\omega \cup \xi^N) = 1,$$

where $|V|$ is the number of primal lattice vertexes of the Dobrushin domain, and $F(\omega \cup \xi^N)$ is the number of faces (including the infinite face) of the graph defined by the configuration. Now we note that since the boundary condition edges ξ^N and the edges of the Dobrushin domain E are disjoint, $\xi^N \cap E = \emptyset$, we have $E(\omega \cup \xi^N) = E(\omega) + |\xi^N|$, no matter what the configuration ω is, because the edges of the configuration ω are subset of the Dobrushin domain edges E.

Now when we simply note that the number of loops in the corresponding configuration $\sigma(\omega)$ is

$$N(\sigma(\omega)) = \Big(F(\omega \cup \xi^N) - 1\Big) + \Big(C(\omega \cup \xi^N) - 1\Big)$$

where it has been noted that each loop in the corresponding configuration can be seen as either encircling a component of original configuration or encircling a component of the dual configuration (the loop encircles the component that is the component "closest" to the loop on the inside of the loop); every dual component except the one corresponding to the dual free arc corresponds to a face of the original graph (the dual component must be surrounded by a component of the original graph, and hence the dual component can be corresponded to the face of a component of the original graph, namely that face that is encircled by those edges who form the closest surrounding of the dual component); thus, there are $F(\omega \cup \xi^N) - 1$ loops encircling a dual component, and the rest of the loops encircle primal component; each of the primal component except the one corresponding to the wired arc are encircled; hence there are $C(\omega \cup \xi^N) - 1$ of these kind of loops (alternatively, each loop can be seen as either encircling a component from the outside or encircling a face from the inside and the infinite face and the component of the wired arc are not encircled).

Now we can eliminate the number of faces from the Euler's formula and we get

$$\begin{aligned}
&|V| - |E(\omega \cup \xi^N)| + F(\omega \cup \xi^N) - C(\omega \cup \xi^N) \\
&= |V| - |E(\omega)| - |\xi^N| + N(\sigma(\omega)) - C(\omega \cup \xi^N) + 2 - C(\omega \cup \xi^N) \\
&= |V| - |E(\omega)| + N(\sigma(\omega)) - 2C(\omega \cup \xi^N) - |\xi^N| + 2 = 1,
\end{aligned}$$

from which we can solve for the number of clusters:

$$C(\omega \cup \xi^N) = \frac{1}{2}\left(|V| - |E(\omega)| + N(\sigma(\omega)) - |\xi^N| + 1\right).$$

Note that if we had looped around the cluster corresponding to the wired arc, we would not have the +1 in the formula for clusters; usually the formula is presented without the +1.

Now we can formulate the probability of the configuration ω as

$$\begin{aligned}
P_{(D,a,b)}(\omega) &= \frac{1}{Z} q^{C(\omega \cup \xi^N)} \prod_{e \in E} p^{\omega(e)} (1-p)^{1-\omega(e)} \\
&= \frac{1}{Z} q^{\frac{1}{2}\left(|V| - |E(\omega)| + N(\sigma(\omega)) - |\xi^N| + 1\right)} p^{|E(\omega)|} (1-p)^{|E| - |E(\omega)|} \\
&= \frac{(1-p)^{|E|} \sqrt{q}^{|V| - |\xi^N| + 1}}{Z} \left(\sqrt{q}\right)^{N(\sigma(\omega))} \left(\frac{p}{(1-p)\sqrt{q}}\right)^{|E(\omega)|} \\
&= \frac{1}{Z'} \left(\frac{p}{(1-p)\sqrt{q}}\right)^{|E(\omega)|} \left(\sqrt{q}\right)^{N(\sigma(\omega))} \\
&= \frac{1}{Z'} \left(\frac{\frac{\sqrt{2}}{1+\sqrt{2}}}{\left(1 - \frac{\sqrt{2}}{1+\sqrt{2}}\right)\sqrt{2}}\right)^{|E(\omega)|} \left(\sqrt{q}\right)^{N(\sigma(\omega))} \\
&= \frac{1}{Z'} (1)^{|E(\omega)|} \left(\sqrt{q}\right)^{N(\sigma(\omega))} = \frac{1}{Z'} \left(\sqrt{q}\right)^{N(\sigma(\omega))},
\end{aligned}$$

where Z' is the modified normalizing constant: $\frac{1}{Z'} = \frac{(1-p)^{|E|}\sqrt{q}^{|V| - |\xi^N| + 1}}{Z}$; also we note that the number of closed edges is exactly $|E| - |E(\omega)|$, and we used the fact that we have chosen parameters $q = 2$, $p = \frac{\sqrt{2}}{1+\sqrt{2}}$. Hence the probability of the configuration depends only on the number of loops present.

Now we are in a position to prove the above flow condition (the proof is formulated before for example in [7], and the basic idea of the proof was presented in [72]):

Proof of Lemma 7.2. Let v and e_1, e_2, e_3 and e_4 be as described in the formulation of the lemma. Now note that the vertex v corresponds to an unique edge of the primal and an unique edge of the dual graph (the edges that intersect at v). Let us define an involution I of the space Ω, that is, a bijection $I : \Omega \to \Omega$ such that the involution of a configuration ω is the configuration $I(\omega)$ in which the state of the primal edge passing through v is different (that is, if the edge was open in the original configuration, it is closed in the

involution and vice versa) and the state of all the other primal edges is the same as in ω.

For an edge e of the medial lattice domain $D_\diamond$, denote

$$\pi_e(\omega) = e^{-\frac{1}{2}iW(\gamma_\omega, e_a, e)} \chi_{e \in \gamma_\omega} P_{(D,a,b)}(\omega)$$

the contribution of the configuration ω to $F(e)$. Through the definition of I we see that the following holds:

$$F(e) = \sum_{\omega \in \Omega} \pi_e(\omega) = \frac{1}{2} \sum_{\omega \in \Omega} (\pi_e(\omega) + \pi_e(I(\omega))),$$

where the sum is finite due to the finiteness of the domain. Now to prove (7.15) one just needs to show that for any configuration ω

$$\begin{aligned} &\pi_{e_1}(\omega) + \pi_{e_1}(I(\omega)) + \pi_{e_3}(\omega) + \pi_{e_3}(I(\omega)) \\ &= \pi_{e_2}(\omega) + \pi_{e_2}(I(\omega)) + \pi_{e_4}(\omega) + \pi_{e_4}(I(\omega)). \end{aligned} \tag{7.16}$$

So, note that when γ_ω does not go though v, nor does $\gamma_{I(\omega)}$. This is because the exploration path goes through v if and only if the configuration is such that at least one of the sites of the primal lattice corresponding to the black diamonds around v (there are two) can be connected to wired arc by an open path of the primal domain, and at least one of the sites of the dual lattice corresponding to the white diamonds around v (there are two) can be connected to the dual free arc by a dual open path of the dual domain. But the existence of these kinds of paths obviously does not depend on the state of the primal edge intersecting v (the state of the primal edge determines also the state of the dual edge intersecting v), because the state of this edge only determines how the exploration path behaves at v (to which direction it turns to continue its journey) but the fact that the exploration path arrives at v is determined by the states of the edges the exploration path has met previously (before meeting v) on its journey. Hence we see that since the involution does not change the states of those edges that determine whether the exploration path meets v or not, so γ_ω meets v if and only if so does $\gamma_{I(\omega)}$. Hence in the case that γ_ω does not meet v, the equation (7.16) reads $0 = 0$.

Thus let γ_ω meet v; that is, the path goes through at least one edge adjacent to v; immediately (due to the way the medial lattice is directed) we see that there are actually only two possibilities: either the path passes through v once using two edges adjacent to v (the path must enter and exit v) or the path passes through v twice using all four edges adjacent to v (the path enters and exits v twice). We see that depending on relation of the vertex v of the medial lattice to the primal lattice we have two options: either the vertex v has a black diamond above and below it or the black diamonds are to the left and right of v (there must be two black and two white diamonds adjacent to v, for v cannot be in the boundary of the domain because it has four medial lattice edges adjacent to it). Let us consider the case where the black diamonds are to the left and right of v; the other case is symmetric.

	e_1	e_2	e_3	e_4
ω	x	$e^{i\frac{\pi}{4}}x$	0	0
$I(\omega)$	$\frac{1}{\sqrt{2}}x$	$\frac{1}{\sqrt{2}}e^{i\frac{\pi}{4}}x$	$\frac{1}{\sqrt{2}}e^{i\frac{\pi}{2}}x$	$\frac{1}{\sqrt{2}}e^{-i\frac{\pi}{4}}x$

TABLE 1. Values $\pi_{e_i}(\omega)$ and $\pi_{e_i}(I(\omega))$, $i = 1, 2, 3, 4$ as multiples of $x = \pi_{e_1}(\omega)$.

Now there are two possibilities for the exploration path to enter the vertex, namely through edges e_1 or e_3, and two possibilities for the path to exit v, namely through e_2 or e_4. Let us assume the path enters v from e_1 and exits the last time through e_2; the other cases are similar. We are left with two possibilities: either e_3 and e_4 are not part of the path at all (the path enters through e_1 and exits through e_2 and never comes back) or they both are part of the path (the path enters through e_1 and exits through e_4 and comes back through e_3, forming a loop in the exploration path, and then exits for the last time through e_2). Now we see that if we have the first case in configuration ω, also e_3 and e_4 are not part of the path, then e_3 and e_4 must be part of a loop because the loop configuration corresponding to the original configuration as presented above is Eulerian, and the primal edge intersecting v must be open. Now in the involution $I(\omega)$, the situation changes so that e_3 and e_4 and the loop they belong to become part of the exploration path; hence we see that our two possible cases are involutions of each other. Now, knowing the term $\pi_{e_1}(\omega)$, we can calculate the contribution of ω and $I(\omega)$ to the values of the observable at all the edges adjacent to v. This will lead us to our claim.

So, let ω correspond to the case that only e_1 and e_2 belong to the exploration path. We note that probability of $I(\omega)$ is $\frac{1}{\sqrt{2}}$ times the probability of ω since there is one loop less in $I(\omega)$. The second thing we note is that the windings of the exploration path (which is a non-self-intersecting path) on the edges around v can be related to the winding $W(\gamma_\omega, e_a, e_1)$. For example, the winding $W(\gamma_\omega, e_a, e_2) = W(\gamma_\omega, e_a, e_1) + \frac{\pi}{2}$. Hence we get that the contributions of ω and $I(\omega)$ to the values of the observable at edges adjacent to v are as presented in table 1.

By summing the contributions as in (7.16) and noticing $e^{i\frac{\pi}{4}} + e^{-i\frac{\pi}{4}} = \sqrt{2}$, $e^{i\frac{\pi}{4}} = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$, $e^{i\frac{\pi}{2}} = i$, we see that the claim holds; the left hand side of (7.16) equals:

$$\begin{aligned} \text{LHS} &= \pi_{e_1}(\omega) + \frac{1}{\sqrt{2}}\pi_{e_1}(\omega) + 0 + \frac{1}{\sqrt{2}}i\pi_{e_1}(\omega) \\ &= \left(1 + \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)\pi_{e_1}(\omega), \end{aligned}$$

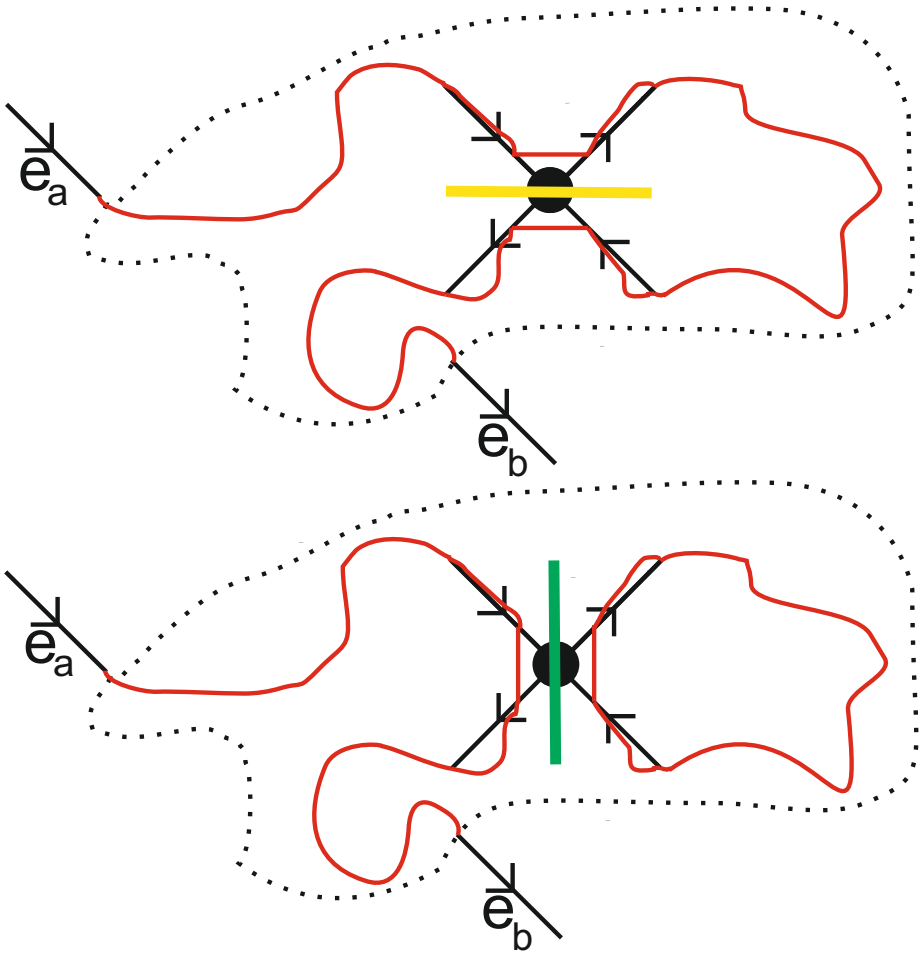


FIGURE 48. The effect of "flipping" (involuting) the state of an edge corresponding to a medial vertex of the exploration path.

and the right hand side of (7.16) equals:

$$\begin{aligned}\text{RHS} &= \frac{1+i}{\sqrt{2}}\pi_{e_1}(\omega) + \frac{1}{\sqrt{2}}\frac{1+i}{\sqrt{2}}\pi_{e_1}(\omega) + 0 + \frac{1}{\sqrt{2}}\frac{1-i}{\sqrt{2}}\pi_{e_1}(\omega)\\ &= \left(\frac{1}{\sqrt{2}} + \frac{1}{2} + \frac{1}{2} + \frac{i}{\sqrt{2}} + \frac{i}{2} - \frac{i}{2}\right)\pi_{e_1}(\omega)\\ &= \left(1 + \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)\pi_{e_1}(\omega).\end{aligned}$$

Hence we can deduce the claim of the Lemma. □

We notice that the complex argument of $F(e)$ follows from the direction of $e \in E_\diamond$ with respect to e_a; for if the edges e, e_a have the same direction, the winding is a multiple of 2π, and so the complex term $e^{-\frac{1}{2}iW(\gamma_\omega, e_a, e)} = \pm 1$ and so $F(e)$ is real (the attribute "*fermionic*" in fermionic observable comes

form the fact that as the winding of the exploration path in the configuration changes by 2π, the weight of the configuration changes to negative). Similarly we can see that the complex argument can be $\pm e^{\frac{i\pi}{4}}$, $\pm e^{\frac{-i\pi}{4}}$, $\pm i$. This property is due to the value of the spin $\frac{1}{2}$. We also note that for a vertex $v \in V_\diamond$ with four adjacent vertexes as in lemma 7.2 the $F(e_1)$ and $F(e_3)$ are always orthogonal (with respect to the usual inner product $Re(a\overline{b})$, where $a, b \in \mathbb{C}$ and $\overline{b}$ is the complex conjugate of b) and so are $F(e_2)$ and $F(e_4)$. Therefore, we get from equation (7.15) by using the Pythagorean theorem to the complex number $F(e_1) + F(e_3)$ to calculate its magnitude, that

$$|F(e_1)|^2 + |F(e_3)|^2 = |F(e_2)|^2 + |F(e_4)|^2. \tag{7.17}$$

Let us then consider a vertex of the medial lattice that is on the boundary, $v \in \partial V_\diamond \subset V_\diamond$. Now v can have two or four edges of $D_\diamond$ adjacent (we do not consider e_a, e_b here). Consider the case with two adjacent edges; see figure 49. Now we see that the other one of the edges is the "entering" and the another the "exiting" edge, denote them e_{in} and e_{out} respectively. Also the vertex v must be located at the boundary of $D_\diamond$ and the exploration path γ_ω must arrive at v through e_{in} and leave through e_{out} if it is to go through v at all. Also, the winding of the path going through e_{in} and e_{out} must be constant, because in a simply connected domain the simple path cannot wind around v and it cannot contain any loops, and the path must always approach v from the same direction (that is, from the direction of e_{in}). Therefore, we can state

$$\begin{aligned} |F(e_{in})|^2 &= \left| e^{-\frac{1}{2}iW(\gamma_\omega, e_a, e_{in})} P_{(D,a,b)}(e_{in} \in \gamma_\omega) \right|^2 \\ &= P_{(D,a,b)}(e_{in} \in \gamma_\omega)^2 \\ &= |F(e_{out})|^2. \end{aligned} \tag{7.18}$$

Then consider the case with four adjacent edges. Now there are two possibilities. Either of the four adjacent edges, only two belong to the Dobrushin boundary arcs ∂_{ab}, ∂_{ba} or all four do (again, we do not consider e_a, e_b here). The case of two edges belonging to the Dobrushin boundary arcs does not interest us (for this case is not relevant in the setting we are about to apply (7.18) and the relatives of it we are about to establish now), so let us study the case of four edges belonging to the Dobrushin boundary arcs. Note that because of the definition of a Dobrushin domain, this means that the boundary path ∂_{ab} or ∂_{ba}, whichever is the one going through v, must go through v two times from opposite directions. Let us say that this path is ∂_{ab} (respectively, ∂_{ba}). Now there are two different ways this can happen: either the dual (primal) edge going through v is part of the domain (D, a, b) or it is not. If not, then we see that v is actually in such a position that the dual (primal) edge going through v is part of the dual free (wired) boundary conditions ξ_d^N (ξ^n), and the exploration path has only two ways it can travel through v, namely it can come from the left/right side of v if the dual (primal) edge going through v is horizontal or up/bottom side of v if the afore mentioned

edge is vertical, and then leave on the same side of v as it entered. Then the edges adjacent to v are naturally divided into two classes: those two on one side of the dual (primal) edge going through v and those on the other, and in each of these classes, one of the vertexes is the "entering" and the other the "exiting" edge, and a similar relation as above in (7.18) can be established between the values of F on these edges, since if the exploration path comes to v, then it must enter through an entering edge of either class, and leave through the exiting edge of the same class. So now, consider the case that the dual (respectively, primal) edge intersecting v is part of the domain $(D, a, b)_d$ ((D, a, b)). Then we see that since D (D_d) must be a simply connected planar domain (or to be precise, a lattice approximation of such a domain) then the only possibility is that v is part of some kind of an isthmus (fjord) in the domain, such that if the exploration path enters the isthmus (fjord) then is must come out of that isthmus (fjord) through v, and also it cannot gather any additional winding on its travels in the isthmus (fjord), so that the winding when the exploration path enters an edge adjacent to v is a constant (depending on the edge in question, of course) for all exploration paths. Now the edges adjacent to v can be divided again into two classes: those that are on the side of the isthmus (fjord) and those that on the side of the domain to which e_a and e_b belong; let us call this the "domain side". In these two classes, the other edge is an "entering" edge and the other is "exiting", and we note that there are two possibilities if the exploration path meets v: either the exploration path enters through the entering vertex of the domain side and exits through the exiting vertex of the domain side, or it enters through the entering domain side edge, continues to exiting isthmus (fjord) side edge, continues and eventually returns through the isthmus (fjord) side entering edge, and exits through the domain side exiting edge. All in all, we see that relation similar to (7.18) can be established between the values of F on entering/exiting isthmus(fjord)/domain side edges.

From equations (7.17), (7.18) we get

Theorem 7.10. *There exist a unique real-valued function H defined on the faces of $D_{\partial\diamond}$ such that*

$$H(B) - H(W) = |F(e)|^2 \tag{7.19}$$

for any two neighbouring black (B) and white (W) faces separated by an edge $e \in E_\diamond$ and such that $H(a) = 1$, where a is the black face corresponding to the vertex a in the primal lattice. Also, H is equal to 1 *on the diamonds of the wired arc and* 0 *on the diamonds of the dual free arc.*

Proof. For this proof, let us define an auxiliary graph $\diamond_D$ whose vertexes are the diamonds of $D_{\partial\diamond}$ and that has edges between adjacent diamonds; diamonds are adjacent if they share an edge, a common vertex is not enough.

One can formulate the function H that is needed as follows: one starts by fixing the value of the function in one boundary medial diamond (as said above, $H(a) = 1$, where a is the black face corresponding to the vertex a

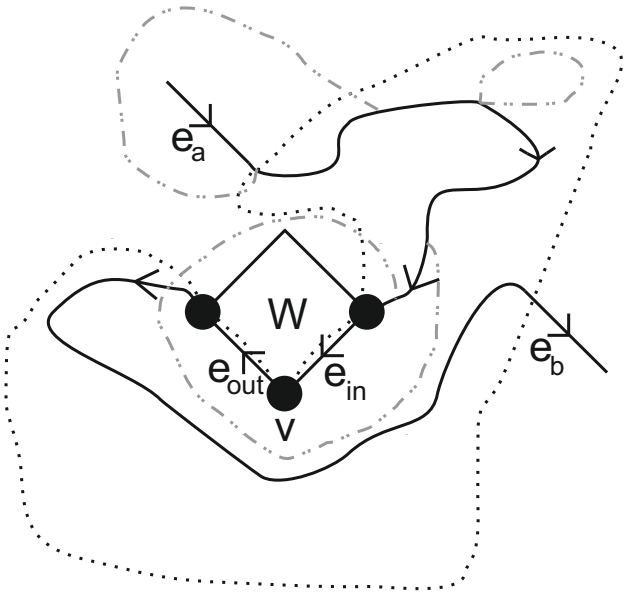


FIGURE 49. The exploration path going through a "boundary" medial vertex. The grey paths show impossible windings of the exploration path.

of the primal lattice) and then one starts to "expand" the function to other diamonds by defining the function such that the increments between adjacent diamonds follow the equation (7.19). Because the Dobrushin domain $D_{\partial\diamond}$ is simply connected (due to the way we defined Dobrushin domains), this construction can be used to define the function throughout the domain. It can be seen that this definition not only produces the needed function but also fixes it uniquely, and that the constructed function is real-valued, because the increments and the "initial value" $H(a) = 1$ are real. What we need to prove is that this construction is consistent and does not lead to contradictions.

Now note that it is enough to show that, in the graph $\diamond_D$, as one goes around a vertex of the domain $D_{\partial\diamond}$ that has 4 adjacent diamonds, let us call these type of vertexes "*interior vertexes*", stepping from a medial diamond to a neighbouring medial diamond, the increments of H add up to zero to show that the above construction is consistent. Note that it is enough to study an interior vertex and the encircling of it, since the boundary vertexes cannot be circled in a simply connected domain. When it has been shown that encircling one interior vertex causes no problems, then if we have two lattice diamonds $A, B \in V(\diamond_D)$, and we have two distinct paths γ^1 and γ^2 (of graph $\diamond_D$) such that both paths connect A and B, we can show that the increment of the function H is be the same when calculated along the path γ^1 and when calculated along the path γ^2, as indeed should be the case for the above construction to be consistent. We show that these increments are the same by transforming path γ^1 to the path γ^2 by making such changes

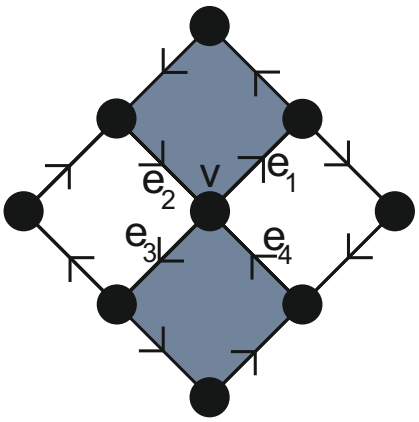


FIGURE 50. The situation around an interior vertex v.

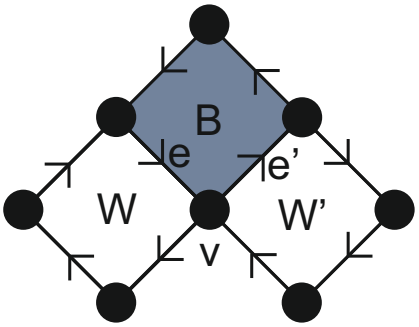


FIGURE 51. The situation on the boundary arcs.

to the path γ^1 that do not change the value of the increment along the path, and that gradually change the path γ^1 into γ^2. These changes to the path that we are referring to are such that we change the path γ^1 locally around an interior vertex such that we approach the path γ^2 bit by bit, one change at a time (one change only concerns the shape of the path in the area of the four diamonds adjacent to the interior vertex). Making these changes around interior vertexes do not change the increment of the whole path because of the above mentioned property that circling an interior vertex, the increment is zero. By changing the shape of the path around many (different) interior vertexes many times, we can shape the path γ^1 to coincide with γ^2. This transformation can be done always since the domain is simply connected.

So, let v be an interior vertex, that is, vertex having 4 edges adjacent to it, and let e_1, e_2, e_3, e_4 be the edges adjacent to v indexed in anti-clockwise order starting from the upper right, as above. The sum of the increments of H as we go around v anti-clockwise starting from the diamond below e_1 is then

$$\pm|F(e_1)|^2 \mp |F(e_2)|^2 \pm |F(e_3)|^2 \mp |F(e_4)|^2,$$

where the upper signs refer to the case where the diamond at which we start is white and the lower signs refer to the case where the starting diamond is black.

Now we can see that using equation (7.17) this reduces to

$$\begin{aligned}
&\pm \left(|F(e_1)|^2 - |F(e_2)|^2 + |F(e_3)|^2 - |F(e_4)|^2 \right) \\
&= \pm \left(\left(|F(e_1)|^2 + |F(e_3)|^2 \right) - \left(|F(e_2)|^2 + |F(e_4)|^2 \right) \right) \\
&= 0.
\end{aligned}$$

To see that the function H is constant of the boundary arcs, let us study two consecutive boundary arc diamonds W and W'. Note that the squares on the boundary arcs are of the same color; that is, they are black on the wired arc and white on the dual free arc; let W and W' be white (the same argument can be applied on the wired arc). Now W and W' have a common vertex v, and it must be the case that v has two or four edges adjacent in $D_\diamond$ (note that e_a, e_b separate the white diamonds of the dual free arc and the black diamonds of the wired arc, so they can not become into consideration here). In the case that there is only two adjacent edges, then we find easily a common adjacent black diamond B of W and W', and when e is the edge separating W and B and e' the edge separating W' and B, we get by (7.19) (which we know that H satisfies)

$$\begin{aligned}
H(W) - H(W') &= (H(W) - H(B)) + (H(B) - H(W')) \\
&= -|F(e)|^2 + |F(e')|^2 = 0,
\end{aligned} \tag{7.20}$$

wherein we have used (7.18); note that e and e' must be on the boundary of $D_\diamond$.

Let us then study the case that v has four adjacent edges in $D_\diamond$; two or four of the edges adjacent to v belong to the Dobrushin boundary arcs ∂_{ab}, ∂_{ba}. Actually we see that since D is a simply connected domain (or actually a lattice approximation of such a domain), it is not possible that for a vertex that has four edges and two boundary arc diamonds adjacent to it, only two of its edges belong to the Dobrushin boundary arcs, since the Dobrushin boundary arcs ∂_{ab}, ∂_{ba} circle the domain on the inside of the wired and dual free arcs (and the two other edges adjacent to v that are not on the Dobrushin boundary arcs, and which should thus belong to the domain, are left out of the domain because of this circling of the inside of the wired and dual free arcs). Thus so let us study the case that v has four edges adjacent in $D_\diamond$ and they all are in the Dobrushin boundary arcs. Now, referring to the studies of this case above, we see that we can find a common adjacent black diamond B of W and W' such that this black diamond belongs to $D_\diamond$ and such that the edges separating B and W, let is be e, and B and W', let it be e', belong to the same class of edges (where by "class" we refer to the classes the edges adjacent to a such a vertex v were divided above), and thus we can carry out a similar argument as in (7.20).

This proves that H is constant on the boundary arcs. This also shows that H is equal to 1 on the diamonds of the wired arc; to see that H is 0 on the dual free arc, note that $F(e_a) = 1$ by (7.14); the rest follows from (7.19). □

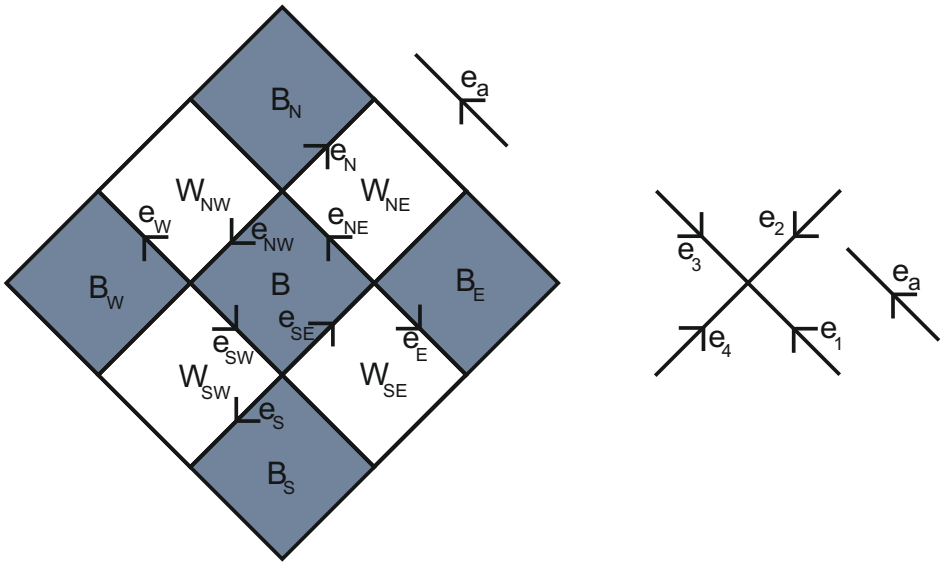


FIGURE 52. The denotations of the proof of Theorem 7.11.

Let us now describe an *approximate Dirichlet problem* for H. Let us denote by $H_\bullet$ the restriction of H to the black diamonds of $D_{\partial\diamond}$ and by $H_\circ$ the restriction to white diamonds. For a black (or white) diamond of $D_{\partial\diamond}$ not on the boundary, we can study the *discrete Laplacian*: the average of the difference of values between the diamond in question and the four neighbouring black (or white) diamonds of it. In denotation of figure 52, the discrete Laplacian for $H_\bullet$ at black diamond B, denoted $\nabla^2 H_\bullet(B)$ is:

$$\nabla^2 H_\bullet(B) = \frac{1}{4}\left(H_\bullet(B_N) + H_\bullet(B_W) + H_\bullet(B_S) + H_\bullet(B_E)\right) - H(B) \tag{7.21}$$

(note how the discrete Laplacian is similar to the mean-value-property that harmonic functions have). Now we claim that:

Theorem 7.11. *The $H_\bullet$ is subharmonic and the $H_\circ$ is superharmonic, meaning that in the domain $D_\diamond$, understood as a domain of diamonds,*

$$\nabla^2 H_\bullet \geq 0 \geq \nabla^2 H_\circ.$$

Proof. We will prove the claim by a direct calculation; by Theorem 7.10, we can relate the differences of H between neighbouring black (white) diamonds to the values of F in the edges separating these diamonds through equation (7.19). Also, we can express the Laplacian in terms of F in the boundary of a diamond of $D_\diamond$.

We shall prove the claim for $H_\bullet$; the claim for $H_\circ$ can be proved similarly. Let B be a black diamond; let the neighbouring black diamonds (those with which B has a common vertex) be B_N, B_W, B_S and B_E, indexed from the upper one, going anti-clockwise. Our strategy is to express $\nabla^2 H_\bullet$ is terms of F and then use the flow condition (7.14) to eliminate the values of F in all other edges except those edges that border B. When we get the Laplacian

expressed in the mentioned form, we shall (after some analysis) see that $\nabla^2 H_\bullet(B) \geq 0$.

By the definition of the discrete Laplacian,

$$\begin{aligned}\nabla^2 H_\bullet(B) &= \frac{1}{4}\left(H_\bullet(B_N) - H_\bullet(B)\right) + \frac{1}{4}\left(H_\bullet(B_W) - H_\bullet(B)\right) \\ &\quad + \frac{1}{4}\left(H_\bullet(B_S) - H_\bullet(B)\right) + \frac{1}{4}\left(H_\bullet(B_E) - H_\bullet(B)\right) \\ &= \frac{1}{4}\left(\partial_N H_\bullet(B) + \partial_W H_\bullet(B) + \partial_S H_\bullet(B) + \partial_E H_\bullet(B)\right),\end{aligned}$$

where we use the denotation $\partial_i H_\bullet(B) = H_\bullet(B_i) - H_\bullet(B)$, $i = N, W, S, E$.

Now let us denote the medial edges bordering B as e_{NE}, e_{NW}, e_{SW} and e_{SE}, starting from the upper right and going anti-clockwise, and the white diamonds across those edges as W_{NE}, W_{NW}, W_{SW} and W_{SE} respectively. Let e_N be the medial edge bordering B_N and W_{NE}, let e_W be the edge bordering B_W and W_{NW}, let e_S be the edge bordering B_S and W_{SW}, and let e_E be the edge bordering B_E and W_{SE}. Then, let us calculate the difference $\partial_N H_\bullet$; by the definition of H, equation (7.19), we have

$$\partial_N H_\bullet(B) = (H(B_N) - H(W_{NE})) - (H(B) - H(W_{NE})) = |F(e_N)|^2 - |F(e_{NE})|^2.$$

Similarly we get the other differences

$$\begin{aligned}\partial_W H_\bullet(B) &= |F(e_W)|^2 - |F(e_{NW})|^2, \\ \partial_S H_\bullet(B) &= |F(e_S)|^2 - |F(e_{SW})|^2, \\ \partial_E H_\bullet(B) &= |F(e_E)|^2 - |F(e_{SE})|^2.\end{aligned}$$

Now let us start to eliminate from the above differences the edges that are not bordering B in favour of edges bordering B. The flow condition (7.14) allows us to express the value of F in one edge utilizing the values of F in the three other adjacent edges; however, noticing as above that the direction of the edge e in relation to the direction of the edge e_a defines the line in complex plane that $F(e)$ belongs to, one can see as above that the values of F in the four adjacent edges form two orthogonal bases in the complex plane in the sense that the values of F in edges that have opposite directions are orthogonal and thus form a base on the complex plane (complex plane is understood as the $\mathbb{R}^2$ vector space) and the two bases hereby formed in the plane are rotated from each other by $\frac{\pi}{4}$; this means that the values of F on whichever two adjacent edges form a (not necessarily orthogonal) base on the complex plane; hence one can express the value of F in some edge e with utilizing only two values of F on adjacent edges; this of course means that the whole Laplacian can be expressed using values of F in the edges bordering B.

To be more concrete, let e_1, e_2, e_3, e_4 be adjacent edges such that e_1 has the same direction as e_a and the edges are listed in anti-clockwise order. Then $F(e_1)$ and $F(e_3)$ are orthogonal as are $F(e_2)$ and $F(e_4)$. Furthermore the basis formed by the lines that $F(e_2)$ and $F(e_4)$ belong to is rotated by

$\frac{\pi}{4}$ with respect to the basis formed by lines that $F(e_1)$ and $F(e_3)$ belong to. To be more precise, we see (as noted above) that $F(e_1)$ corresponds to line $\mathbb{R}$, $F(e_2)$ corresponds to line $e^{i\frac{\pi}{4}}\mathbb{R}$, $F(e_3)$ corresponds to line $i\mathbb{R}$ and $F(e_4)$ corresponds to line $e^{i\frac{3\pi}{4}}\mathbb{R} = e^{-i\frac{\pi}{4}}\mathbb{R}$.

Let us now define $\widehat{F(e_i)} \in \mathbb{R}$, $i = 1, 2, 3, 4$ such that $F(e_i) = \widehat{F(e_i)}d_i$ where $d_1 = 1$, $d_2 = e^{i\frac{\pi}{4}}$, $d_3 = i$ and $d_4 = e^{i\frac{3\pi}{4}}$; also we fix the positive direction in the lines that $F(e_i)$ belong to. By the flow relation (7.14) $F(e_1) + F(e_3) = F(e_2) + F(e_4)$ we can view the values $\widehat{F(e_i)}$ as the components of the vector $A = F(e_1)+F(e_3) = F(e_2)+F(e_4)$ to the corresponding lines d_i, $i = 1, 2, 3, 4$ (here we use the vectors d_i to define an oriented line).

Now we use simple linear algebra: if a vector

$$v = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} |v|\cos(\phi) \\ |v|\sin(\phi) \end{bmatrix},$$

where ϕ is the phase angle of the vector and $|v|$ is the magnitude or length of the vector, is rotated by angle α, then its new coordinates are

$$\begin{aligned} \text{Rotation}_\alpha(v) &= \begin{bmatrix} |v|\cos(\phi+\alpha) \\ |v|\sin(\phi+\alpha) \end{bmatrix} = \begin{bmatrix} |v|\left(\cos(\phi)\cos(\alpha) - \sin(\phi)\sin(\alpha)\right) \\ |v|\left(\sin(\phi)\cos(\alpha) + \cos(\phi)\sin(\alpha)\right) \end{bmatrix} \\ &= \begin{bmatrix} x\cos(\alpha) - y\sin(\alpha) \\ y\cos(\alpha) + x\sin(\alpha) \end{bmatrix} = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \end{aligned}$$

since $x = |v|\cos(\phi)$, $y = |v|\sin(\phi)$.

Now we note that if the vector A defined above has coordinates $\begin{bmatrix} \widehat{F(e_1)} \\ \widehat{F(e_2)} \end{bmatrix}$ in the basis defined by the lines d_1 and d_3, and coordinates $\begin{bmatrix} \widehat{F(e_2)} \\ \widehat{F(e_4)} \end{bmatrix}$ in the basis defined by lines d_2 and d_4, and since the latter basis can be reached from the formed by rotating the basis by $\frac{\pi}{4}$, then we can relate the two coordinates by noticing that making a $-\frac{\pi}{4}$ rotation to the vector A in d_1, d_3-basis changes the coordinates of the vector A to $\begin{bmatrix} \widehat{F(e_2)} \\ \widehat{F(e_4)} \end{bmatrix}$; hence on the basis of what we just showed about rotations, we have

$$\begin{aligned} \begin{bmatrix} \widehat{F(e_2)} \\ \widehat{F(e_4)} \end{bmatrix} &= \begin{bmatrix} \cos(-\frac{\pi}{4}) & -\sin(-\frac{\pi}{4}) \\ \sin(-\frac{\pi}{4}) & \cos(-\frac{\pi}{4}) \end{bmatrix} \begin{bmatrix} \widehat{F(e_1)} \\ \widehat{F(e_3)} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \widehat{F(e_1)} \\ \widehat{F(e_3)} \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{\sqrt{2}}\widehat{F(e_1)} + \frac{1}{\sqrt{2}}\widehat{F(e_3)} \\ -\frac{1}{\sqrt{2}}\widehat{F(e_1)} + \frac{1}{\sqrt{2}}\widehat{F(e_3)} \end{bmatrix}. \end{aligned} \tag{7.22}$$

This relation now allows us to solve any two values of F as a function of the other two.

Let us now go back to our original computation of the Laplacian and let us assume that e_{NE} is in the same direction as e_a (the cases where e_a is in the direction of e_{NW}, e_{SW} or e_{SE} are similar).

Because of our choice of the direction of e_a, we again know the lines to which the values $F(e_i)$, $i = N, NW, W, SW, S, SE, E, NE$ belong to. Let us

define $\widehat{F(e_i)} \in \mathbb{R}$ so that $F(e_i) = \widehat{F(e_i)}d_i$ for $i = N, NW, W, SW, S, SE, E, NE$ where $d_{NE} = d_W = d_1$, $d_N = d_{SE} = d_2$, $d_{SW} = d_E = d_3$ and $d_{NW} = d_S = d_4$ where d_i, $i = 1, 2, 3, 4$ are as above. Now it is obvious that we need only to determine $\widehat{F(e_i)}$ to solve $F(e_i)$.

We want first to solve $F(e_N)$ as a function of $F(e_{NE})$ and $F(e_{NW})$, also we want to solve $\widehat{F(e_N)}$ as a function of $\widehat{F(e_{NE})}$ and $\widehat{F(e_{NW})}$. Now this can be done by solving the equivalent quantity to $\widehat{F(e_N)}$, namely $\widehat{F(e_2)}$ using the equivalent quantities to $\widehat{F(e_{NE})}$ and $\widehat{F(e_{NW})}$, namely $\widehat{F(e_1)}$ and $\widehat{F(e_4)}$ respectively. From the above equations (7.22) we get

$$\widehat{F(e_3)} = \sqrt{2}\widehat{F(e_4)} + \widehat{F(e_1)}$$

and using the $\widehat{F(e_2)}$-equation of (7.22)

$$\begin{aligned}\widehat{F(e_2)} &= \frac{1}{\sqrt{2}}\left(\widehat{F(e_1)} + \widehat{F(e_3)}\right) = \frac{1}{\sqrt{2}}\left(\widehat{F(e_1)} + \sqrt{2}\widehat{F(e_4)} + \widehat{F(e_1)}\right)\\ &= \sqrt{2}\widehat{F(e_1)} + \widehat{F(e_4)},\end{aligned}$$

or in the similar case, $\widehat{F(e_N)} = \sqrt{2}\widehat{F(e_{NE})} + \widehat{F(e_{NW})}$. Next we want to solve $F(e_W)$ as a function of $F(e_{NW})$ and $F(e_{SW})$; as above we see that it is enough to solve $\widehat{F(e_1)}$ as a function of $\widehat{F(e_3)}$ and $\widehat{F(e_4)}$. Again we utilize equations (7.22) to get

$$\widehat{F(e_1)}) = \frac{1}{\sqrt{2}}\left(\widehat{F(e_2)} - \widehat{F(e_4)}\right),$$

$$\begin{aligned}\widehat{F(e_2)}) &= \frac{1}{\sqrt{2}}\left(\widehat{F(e_1)} + \widehat{F(e_3)}\right) = \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\left(\widehat{F(e_2)} - \widehat{F(e_4)}\right) + \widehat{F(e_3)}\right)\\ &= \frac{1}{2}\widehat{F(e_2)} - \frac{1}{2}\widehat{F(e_4)} + \frac{1}{\sqrt{2}}\widehat{F(e_3)},\end{aligned}$$

$$\widehat{F(e_2)} = \sqrt{2}\widehat{F(e_3)} - \widehat{F(e_4)},$$

$$\begin{aligned}\widehat{F(e_1)} &= \frac{1}{\sqrt{2}}\left(\widehat{F(e_2)} - \widehat{F(e_4)}\right) = \frac{1}{\sqrt{2}}\left(\sqrt{2}\widehat{F(e_3)} - \widehat{F(e_4)} - \widehat{F(e_4)}\right)\\ &= \widehat{F(e_3)} - \sqrt{2}\widehat{F(e_4)},\end{aligned}$$

or in the similar case, $\widehat{F(e_W)} = \widehat{F(e_{SW})} - \sqrt{2}\widehat{F(e_{NW})}$. Next we move on to solve $F(e_S)$ as a function of $F(e_{SW})$ and $F(e_{SE})$; again it is enough to solve $\widehat{F(e_4)}$ as a function of $\widehat{F(e_3)}$ and $\widehat{F(e_2)}$. Again from equations (7.22) we get

$$\widehat{F(e_1)} = \frac{1}{\sqrt{2}}\left(\widehat{F(e_2)}) - \widehat{F(e_4)}\right),$$

$$\begin{aligned}&\widehat{F(e_4)} = \frac{1}{\sqrt{2}}\left(-\frac{1}{\sqrt{2}}\left(\widehat{F(e_2)} - \widehat{F(e_4)}\right) + \widehat{F(e_3)}\right) = \frac{1}{2}\widehat{F(e_4)} - \frac{1}{2}\widehat{F(e_2)} + \frac{1}{\sqrt{2}}\widehat{F(e_3)}\\ &\Leftrightarrow \widehat{F(e_4)} = \sqrt{2}\widehat{F(e_3)} - \widehat{F(e_2)},\end{aligned}$$

or in the similar case, $\widehat{F(e_S)} = \sqrt{2}\widehat{F(e_{SW})} - \widehat{F(e_{SE})}$. Last we shall write $F(e_E)$ as a function of $F(e_{SE})$ and $F(e_{NE})$; this corresponds to solving $\widehat{F(e_3)}$ as a function of $\widehat{F(e_2)}$ and $\widehat{F(e_1)}$. Again from the equations (7.22) we get

$$\widehat{F(e_2)} = \frac{1}{\sqrt{2}}\left(\widehat{F(e_1)} + \widehat{F(e_3)}\right) \Leftrightarrow \widehat{F(e_3)} = \sqrt{2}\widehat{F(e_2)} - \widehat{F(e_1)},$$

or in the similar case, $\widehat{F(e_E)} = \sqrt{2}\widehat{F(e_{SE})} - \widehat{F(e_{NE})}$. Hence we get by noting that $|d_i|^2 = 1$ for all $i = 1, 2, 3, 4, N, NW, W, SW, S, SE, E, NE$:

$$\begin{aligned}
|F(e_N)|^2 &= \left(\sqrt{2}\widehat{F(e_{NE})} + \widehat{F(e_{NW})}\right)^2 \\
&= 2\widehat{F(e_{NE})}^2 + 2\sqrt{2}\widehat{F(e_{NE})}\widehat{F(e_{NW})} + \widehat{F(e_{NW})}^2, \\
|F(e_W)|^2 &= 2\widehat{F(e_{NW})}^2 - 2\sqrt{2}\widehat{F(e_{NW})}\widehat{F(e_{SW})} + \widehat{F(e_{SW})}^2, \\
|F(e_S)|^2 &= 2\widehat{F(e_{SW})}^2 - 2\sqrt{2}\widehat{F(e_{SW})}\widehat{F(e_{SE})} + \widehat{F(e_{SE})}^2, \\
|F(e_E)|^2 &= 2\widehat{F(e_{SE})}^2 - 2\sqrt{2}\widehat{F(e_{SE})}\widehat{F(e_{NE})} + \widehat{F(e_{NE})}^2.
\end{aligned}$$

Now we can express the Laplacian as

$$\begin{aligned}
\nabla^2 H_\bullet(B) &= \frac{1}{4}\left(\partial_N H_\bullet(B) + \partial_W H_\bullet(B) + \partial_S H_\bullet(B) + \partial_E H_\bullet(B)\right) \\
&= \frac{1}{4}\Big(\widehat{F(e_{NE})}^2 + 2\sqrt{2}\widehat{F(e_{NE})}\widehat{F(e_{NW})} + \widehat{F(e_{NW})}^2 \\
&\quad + \widehat{F(e_{NW})}^2 - 2\sqrt{2}\widehat{F(e_{NW})}\widehat{F(e_{SW})} + \widehat{F(e_{SW})}^2 \\
&\quad + \widehat{F(e_{SW})}^2 - 2\sqrt{2}\widehat{F(e_{SW})}\widehat{F(e_{SE})} + \widehat{F(e_{SE})}^2 \\
&\quad + \widehat{F(e_{SE})}^2 - 2\sqrt{2}\widehat{F(e_{SE})}\widehat{F(e_{NE})} + \widehat{F(e_{NE})}^2\Big).
\end{aligned}$$

Let us now denote $a = \widehat{F(e_{NE})}$, $b = \widehat{F(e_{NW})}$, $c = \widehat{F(e_{SW})}$ and $d = \widehat{F(e_{SE})}$, $a, b, c, d \in \mathbb{R}$. Now we write the Laplacian in terms of these real variables:

$$2\nabla^2 H_\bullet(B) = a^2 + b^2 + c^2 + d^2 + \sqrt{2}ab - \sqrt{2}bc - \sqrt{2}cd - \sqrt{2}da.$$

Now it is obvious that our claim boils down to showing that the quadratic form of four variables a, b, c and d on the right hand side of the above equation is non-negative. We make a note (the inspiration to this idea came from [18]):

$$\begin{aligned}
&a^2 + b^2 + c^2 + d^2 + \sqrt{2}ab - \sqrt{2}bc - \sqrt{2}cd - \sqrt{2}da \\
&= \left(\frac{1}{\sqrt{2}}a + b - \frac{1}{\sqrt{2}}c\right)^2 + \left(d - \frac{1}{\sqrt{2}}a - \frac{1}{\sqrt{2}}c\right)^2
\end{aligned}$$

from which the non-negativity is imminent. □

Now since H is equal to 1 in the wired arc and 0 in the dual free arc, the above Theorem can be seen as a form of discrete Dirichlet boundary value problem.

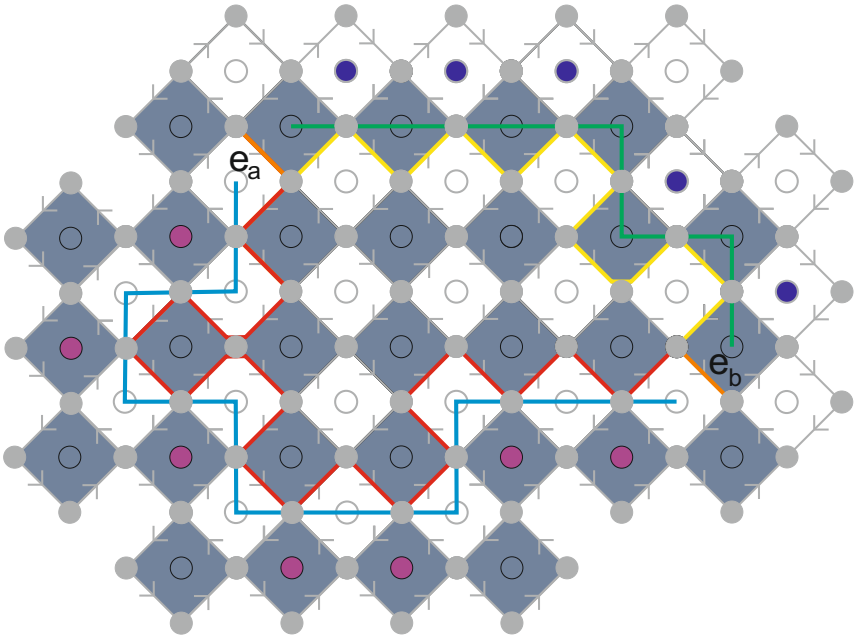


FIGURE 53. The domain $\overline{D_{\partial\diamond}}$. The set $\overline{\partial_{ab}}$ is marked by purple dots and the set $\overline{\partial_{ba}}$ with blue dots.

7.4.5. *Discrete harmonic measures and random walks.* Now let us extend further the medial lattice domain $D_{\partial\diamond}$ in the medial lattice by adding two sets of lattice diamonds (in here, the "diamond" comprises of the medial lattice vertexes and edges of the diamond) into the domain; we shall refer to these sets of diamonds as "additional" or "extra layers" of diamonds. The quotation marks are used above because the geometric structure of these sets of diamonds does not necessarily resemble a "layer" if the domain D has irregularly shaped boundary with "slits" and "fjords" and "spikes" and "isthmuses". Let us define the first set, denote it $\overline{\partial_{ab}}$, which we shall call "extra layer of black diamonds adjacent to the dual free arc", as follows: for a black diamond B (we shall not take any white diamonds into $\overline{\partial_{ab}}$) of medial lattice $B \in \overline{\partial_{ab}}$ if and only if for some free arc black diamond B', B is a neighbouring (sharing a vertex) diamond of B' in the medial lattice behind the borderline ∂_{ab}, id est, the primal lattice edge between B' and B must cross the medial lattice path ∂_{ab} at the vertex common to B' and B at least once. Now let us recall that a primal lattice edge crosses a medial lattice path at the medial lattice vertex where they meet if the medial lattice path is such that it approaches the vertex from one side of the edge and exits the vertex on the other side of the edge. Let us also note that it can be the case that the path ∂_{ab} goes through the vertex between B' and B twice; in that case, the edge between B' and B crosses that path twice, and that is why the "at least once"-condition is added above. Note also that the black diamonds of the set $\overline{\partial_{ab}}$ are adjacent to (have a common edge with) some

white diamonds of the dual free arc, and the in areas where the dual free arc is "straight", they form an additional layer of black diamonds next to the dual free arc just outside the domain $D_{\partial\diamond}$. But as noted above, in areas where the dual free arc is not "straight" or indeed, where the dual free arc is "twisted", $\overline{\partial_{ab}}$ does not resemble a geometrical arc (it is actually quite hard to give any geometrical characterization of the extra layer in areas where the dual free arc is "irregular"). Also note that some medial lattice diamonds B that belong to $\overline{\partial_{ab}}$ also belong to $D_{\partial\diamond}$; this happens for example if the domain has a slit such that the wired and dual free arcs are adjacent along the slit; then the diamonds of the wired arc along the slit belong to $\overline{\partial_{ab}}$. However, we want the set $\overline{\partial_{ab}}$ to be completely disjoint from the domain $D_{\partial\diamond}$, so we demand that if a diamond B belongs to the extra layer and to the domain, then it is considered that the extra layer has a completely separate "copy" the diamond B that is different from the diamond B of the domain (one can for example demand that the extra layer contains ordered pairs $(B, 0) \neq B$ where B is a medial lattice diamond that meets the above presented condition, and when one talks about "extra layer diamonds" one means actually these ordered pairs). The other set of medial lattice diamonds, denoted $\overline{\partial_{ba}}$ and called "extra layer of white diamonds adjacent to the wired arc", contains those white medial lattice diamonds that fulfil a similar condition as above with respect to dual wired arc diamonds, dual edges and ∂_{ba}; similar notes as above also apply to this extra layer of diamonds. Finally, note that the construction of the extra layers of diamonds is completely determined by (D, a, b), so there is no need to spend any additional thought as to what the "extra layers" should be.

Now let us denote the extended domain of (D, a, b) with the extra layers $\overline{\partial_{ab}}, \overline{\partial_{ba}}$ added (as noted above, if some diamonds that are added are already part of the domain, we consider the added diamonds to be disjoint) $\overline{D_{\partial\diamond}}$. Let us define what we mean by "neighbouring diamonds" in this extended domain. Usually we mean by neighbouring diamonds two medial lattice diamonds of the same color that share a vertex, but in context of the extended domain, we want to define for diamonds $B \in \overline{\partial_{ab}}$ (respectively, $W \in \overline{\partial_{ba}}$) that only those diamonds that belong to the original domain $D_{\partial\diamond}$ and that share a medial lattice vertex with B (W) and that are behind the borderline ∂_{ab} (∂_{ba}) when looking from B (W) in the sense of the above definition of the extra layer are neighbouring diamonds of B (W). For black (respectively, white) diamonds B (W) of the original domain $D_{\partial\diamond}$, we define that the neighbours of B (W) are those black (white) diamonds $C \in \overline{D_{\partial\diamond}}$ that share a primal (dual) lattice edge with B (W) when both B (W) and C are understood as vertexes of the primal (dual) lattice, such that if the primal (dual) lattice edge between B and C does not cross the Dobrushin boundary arc ∂_{ab} (∂_{ba}), then C is just the diamond of the domain $D_{\partial\diamond}$, but if the edge crosses the Dobrushin boundary arc, then C is an extra layer diamond

of $\overline{\partial_{ab}}$ ($\overline{\partial_{ab}}$). So note that this leads to following: no two extra layer diamonds (belonging to $\overline{\partial_{ab}} \cup \overline{\partial_{ba}}$) are neighbours, so that extra layer diamonds have only neighbours in the original domain $D_{\partial\diamond}$, and extra layer diamonds have four or less neighbours. Note also that every diamond C of the original domain $D_{\partial\diamond}$ has exactly four neighbours, some of them on the original domain and some on the extra layers, and what decides whether the neighbour is taken to be a diamond or the original domain or of the extra arc is whether or not the neighbour is behind the borderline when looked from C. Of course the "behind the borderline"- and the "edge crossing medial lattice path"-descriptions above must be understood to mean the same thing as in the definition of the extra layers.

For any given black (white) diamond, let us define a *random walk* $(X_n^B)_{n\in\mathbb{N}}$ (respectively, $(Y_n^W)_{n\in\mathbb{N}}$) on black (white) diamonds of $\overline{D_{\partial\diamond}}$ that starts at B (W) and jumps at neighbouring black (white) diamonds with the transition probabilities defined as follows: the random walk connects with the diamonds in the $\overline{\partial_{ab}}$ ($\overline{\partial_{ba}}$) a rate-parameter $\rho = \frac{2}{\sqrt{2}+1}$, and with the diamonds of $D_{\partial\diamond}$ the walk connects the rate-parameter $\rho = 1$. Let us denote the rate connected to a diamond A by $r(A)$. Then the probability of the random walk jumping onto a neighbouring diamond A of the diamond C the random walk currently occupies is the proportion of the rate connected to A to the sum of all rates of the neighbouring diamonds of the diamond C. Note that the walk is more likely to step on to diamonds not in the extra layers, and that the probability of the walk stepping onto a diamond in the extra layer is increased as the number of neighbouring diamonds of the diamond occupied by the walk that are on the extra layers increases. For a black (white) diamond B (W), we denote by HM(B) (HM(W)) the probability that X_n^B (Y_n^W) hits the wired arc (the additional layer of diamonds adjacent to wired arc $\overline{\partial_{ba}}$) before hitting the extra layer adjacent to the dual free arc $\overline{\partial_{ab}}$ (the dual free arc); we call these quantities the *harmonic measure* of the wired arc. We prove:

Theorem 7.12. *Let (D, a, b) be a Dobrushin domain. For any medial edge e of $D_\diamond$ that borders a white diamond of the dual free arc of $D_{\partial\diamond}$, let B be the black diamond e borders and let W be any of the white diamonds not on the dual free arc and adjacent to B (assuming such W exist; if no such diamonds exist, the situation is trivial, as $HM(W) = 0$ for all adjacent white diamonds). Then*

$$\sqrt{HM(W)} \leq |F(e)| \leq \sqrt{HM(B)}.$$

Proof. Let us agree that during this proof, if A is a diamond of $\overline{D_{\partial\diamond}}$, we denote the neighbouring diamonds of A such that the neighbour to the right of A is A_E, the neighbour above A is A_N, the neighbour left of A is A_W, and the neighbour below A is A_S (when we again just consider all the neighbouring diamonds of A as corresponding to vertexes of the primal or dual lattice); if A has less than four neighbours, we keep the denotation but in that case some of the neighbours simply do not exist.

By the definition of H, as stated in Theorem 7.10, we have

$$H(B) = |F(e)|^2 + H(W') = |F(e)|^2 + 0 = |F(e)|^2,$$

where W' is the white diamond on the other side of e from B; note that W' is on the dual free arc of $D_{\partial\diamond}$ and by the definition of H, $H(W') = 0$. Also by the definition of H,

$$H(W) = H(B) - |F(e')|^2 = |F(e)|^2 - |F(e')|^2 \leq |F(e)|^2,$$

where e' is the medial edge between B and W. This is why it suffices to show $H(B) \leq \mathrm{HM}(B)$ and $H(W) \geq \mathrm{HM}(W)$. We show $H(B) \geq \mathrm{HM}(B)$, since the other inequality is proved similarly.

First we extend the function H to the extra layer of black diamonds in $\overline{D_{\partial\diamond}}$ and set $H = 0$ in these diamonds (to extend H to the extra layer of white diamonds we set $H = 1$ on these diamonds); note that here it is crucial that the extra layers are considered separate from the original domain $D_{\partial\diamond}$, for else we could be now giving H two non-equal values on the same diamond. It is then sufficient to show that the restriction $H_\bullet$ of H to the black diamonds of $\overline{D_{\partial\diamond}}$ is subharmonic for the Laplacian that is the generator of X^B. This is because $H_\bullet$ has the same boundary values as $\mathrm{HM}(\bullet)$ (0 in the extra layer, 1 in the wired arc), and $\mathrm{HM}(\bullet)$ is harmonic for the above-mentioned Laplacian (meaning that its Laplacian is zero), so if $H_\bullet$ is subharmonic for the Laplacian, then $f = H_\bullet - \mathrm{HM}(\bullet)$ is also subharmonic for the Laplacian in question, and since the function f has boundary-values 0, it must be non-positive by the maximum principle for discrete harmonic functions presented in the appendix Lemma A.1.

Let us argue why the $\mathrm{HM}(\bullet)$ is harmonic. The discrete Laplacian for the $\mathrm{HM}(\bullet)$ in a face B is:

$$\begin{aligned}
&\nabla^2 \mathrm{HM}(B) \\
&= \frac{r(B_N)\mathrm{HM}(B_N) + r(B_W)\mathrm{HM}(B_W) + r(B_S)\mathrm{HM}(B_S) + r(B_E)\mathrm{HM}(B_E)}{r(B_N) + r(B_W) + r(B_S) + r(B_E)} \\
&\quad - \mathrm{HM}(B).
\end{aligned}$$

Now we see that the above quantity is zero simply by using probability theory; for a random walker who starts at B, there are exactly four options, namely to move to B_N, B_W, B_S or B_E. The probability of moving to B_i is $\frac{r(B_i)}{r(B_N)+r(B_W)+r(B_S)+r(B_E)}$, $i = N, W, S, E$. The movement of the random walker to neighbouring diamonds is completely independent of everything, since the random walker "remembers no history". Thus we arrive at the formula

$$\begin{aligned}
\mathrm{HM}(B) = {} & P(\text{Step at } B_N)\mathrm{HM}(B_N) + P(\text{Step at } B_W)\mathrm{HM}(B_W) \\
& + P(\text{Step at } B_S)\mathrm{HM}(B_S) + P(\text{Step at } B_E)\mathrm{HM}(B_E)
\end{aligned}$$

from which it is easy to see that the Laplacian above is zero.

Let is then study H_{bullet}. Now we note that inside $\overline{D_{\partial\diamond}}$ the subharmonicity of $H_\bullet$ is given by (7.11), since the Laplacian is the usual discrete Laplacian.

Now we only need to check the case in which a diamond that is used in the computation belongs to the extra layers of $\overline{D_{\partial\diamond}}$.

Let us study the case when only one diamond belongs to these extra layers. We are interested of the diamond B and we assume that B_E is in the extra layer. We want to show that:

$$\begin{aligned}(7.23)\quad &\nabla^2 H_\bullet(B)\\ &= \frac{1}{3+\rho}\left(H_\bullet(B_N) + H_\bullet(B_W) + H_\bullet(B_S)\right) + \frac{\rho}{3+\rho}H_\bullet(B_E) - H_\bullet(B)\\ &\geq 0.\end{aligned}$$

Now, let us denote the four edges of the medial lattice $\overline{D_{\partial\diamond}}$ adjacent to the vertex common to B and B_E by e_1, e_2, e_3, e_4 in anti-clockwise order with e_1 and e_2 along B. Here one understands that e_1 and e_2 are edges of B and e_3 and e_4 are edges of B_E; note that this is important if the extra layer diamond B_E is such that there is a diamond B'_E of $D_{\partial\diamond}$ that corresponds to the same vertex of primal lattice, since in this case the function F might be already defined on the edges of B'_E that correspond to same edges in the medial lattice as e_3 and e_4 (note that obviously e_3 and e_4 correspond to unique medial lattice edges). We will extend F to e_3 and e_4 requiring that $F(e_1)$ and $F(e_3)$ are orthogonal, and $F(e_2)$ and $F(e_4)$ also. This gives us 4 different configurations as to the values of $F(e_3)$ and $F(e_4)$. We fix the values by also demanding $F(e_1)-F(e_2) = F(e_4)-F(e_3)$; we see that this sets the values $F(e_4)$ and $-F(e_3)$ uniquely, since those values are the projections of the vector $F(e_1) - F(e_2)$ to the lines orthogonal to $F(e_1)$ and $F(e_2)$, that span the plane. Note that $F(e_2) = e^{\frac{-i\pi}{4}}F(e_1)$ by the definition of F, since every path contributing to the values of $F(e_1)$, $F(e_2)$, must go through e_2 and also through e_1, and the change of the winding $W_\gamma(e_a, e)$ when a path goes from e_2 to e_1 is $\frac{\pi}{2}$, or $W_\gamma(e_a, e_1) = W_\gamma(e_a, e_2) + \frac{\pi}{2}$, so the vectors $F(e_1)$ and $F(e_2)$ are not linearly dependent and therefore the lines orthogonal to those vectors do span the plane. Also from the above we can deduce that

$$\begin{aligned}F(e_1) &\in \mathbb{C}\\ F(e_2) &= e^{\frac{-i\pi}{4}}F(e_1)\\ F(e_3) &= iF(e_1)\cdot a\\ F(e_4) &= iF(e_2)\cdot b,\end{aligned}$$

for some parameters $a, b \in \mathbb{R}$, which we can solve as follows:

$$\begin{cases} F(e_1) - F(e_2) = \left(1 - \dfrac{1-i}{\sqrt{2}}\right)F(e_1) = \left(\left(1 - \dfrac{1}{\sqrt{2}}\right) + \dfrac{i}{\sqrt{2}}\right)F(e_1)\\ F(e_4) - F(e_3) = \left(ia + ib\dfrac{1-i}{\sqrt{2}}\right)F(e_1) = \left(\dfrac{b}{\sqrt{2}} + i\left(a + \dfrac{b}{\sqrt{2}}\right)\right)F(e_1)\end{cases}$$

$$\Leftrightarrow \begin{cases} \dfrac{b}{\sqrt{2}} = 1 - \dfrac{1}{\sqrt{2}} \\ \dfrac{1}{\sqrt{2}} = a + \dfrac{b}{\sqrt{2}} \end{cases}$$

$$\Leftrightarrow a = b = \sqrt{2} - 1,$$

so we get that

$$|F(e_3)|^2 = |F(e_4)|^2 = \left|\left(\sqrt{2}-1\right) e^{\frac{i\pi}{4}} F(e_1)\right|^2 = \frac{2-\sqrt{2}}{2+\sqrt{2}} |F(e_2)|^2 = \frac{2-\sqrt{2}}{2+\sqrt{2}} H_\bullet(B),$$

where $\sqrt{2}-1 = \sqrt{\left(\sqrt{2}-1\right)^2} = \sqrt{\frac{(2+\sqrt{2})(\sqrt{2}-1)^2}{2+\sqrt{2}}} = \sqrt{\frac{\sqrt{2}(\sqrt{2}-1)}{2+\sqrt{2}}} = \sqrt{\frac{2-\sqrt{2}}{2+\sqrt{2}}}$.

Now we define $\overline{H_\bullet}$ to be the function defined on B, B_N, B_W,B_S, B_E as $\overline{H_\bullet} = H_\bullet$ on B, B_N, B_W, B_S and by

$$\overline{H_\bullet}(B_E) = |F(e_3)|^2 = \frac{2-\sqrt{2}}{2+\sqrt{2}} H_\bullet(B)$$

on B_E. Now $\overline{H_\bullet}$ satisfies the relation defining H, (7.10), for e_3 and e_4 as well as for edges that are adjacent to a medial lattice vertex of B (note that B_E is behind the dual free arc as seen from B). Since the fermionic observable F verifies the same local equations around B (F satisfies the flow relation at the vertexes of B, that is, for edges of medial lattice adjacent to a medial lattice vertex of B, and also the complex arguments of the values of F at different edges around vertexes if B behave as was noted above after the proof of the flow relation Lemma 7.2) as above in the proof of Theorem 7.11, we can do again the calculation of the normal discrete Laplacian for H at interior diamond with four neighbours done in Theorem 7.11, but this time do it for $\overline{H_\bullet}$ at the above mentioned diamond B having the extra layer neighbour B_E, and we can deduce as in the Theorem 7.11, that for $\overline{H_\bullet}$ the normal discrete Laplacian at B is non-negative:

$$\nabla^2 \overline{H_\bullet}(B) = \frac{1}{4}\left(\overline{H_\bullet}(B_N) + \overline{H_\bullet}(B_W) + \overline{H_\bullet}(B_S) + \overline{H_\bullet}(B_E)\right) - \overline{H_\bullet}(B) \geq 0.$$

Note that $\overline{H_\bullet}$ need not be defined in other diamonds except at B, B_E, B_N, B_W, B_S and F need not be defined in other edges except those that belong to the afore mentioned diamonds and that are adjacent to a vertex of B for us to be able to repeat the argument and the calculation of the normal discrete Laplacian for $\overline{H_\bullet}$ at B that was done in the proof of Theorem 7.11.

Using the definition of $\overline{H_\bullet}$, we get that

$$\frac{1}{4}\left(H_\bullet(B_N) + H_\bullet(B_W) + H_\bullet(B_S)\right) - \frac{6+5\sqrt{2}}{4\left(2+\sqrt{2}\right)} H_\bullet(B) \geq 0.$$

Since $H_\bullet(B_E) = 0$, we can easily modify this to show the claim (7.23).

The case where two or more neighbours of B belong to the extra layer, we extend the function F around the vertexes common to B and the extra layer vertexes as above, and similarly as above, we get the claim (7.23). □

Let us now consider only Dobrushin domains (D, a, b) such that D contains the origin on the free arc, and $D_\diamond \subseteq \mathbb{H}_\diamond$, where $\mathbb{H}_\diamond$ is the medial lattice domain of the upper half plane. In this case, we say that (D, a, b) is a *Dobrushin* $\mathbb{H}$*-domain*. Let B_0 be the black diamond corresponding to the origin in the medial lattice, and let W_0 be an adjacent white diamond not on the dual free arc of $D_{\partial\diamond}$. Let us also denote $L_n(k) = \{k\} \times ([0, n] \cap \mathbb{Z})$ for *segments*. We call $\mathrm{HM}(B_0)$ ($\mathrm{HM}(W_0)$) the harmonic measure of the wired arc. We denote constants depending on β (our aspect ratio parameter defined in the beginning of this subsection) by $c_i(\beta)$, and independent constants just by c_i.

Lemma 7.3. *Let R_n^β, where $\beta > 0$ and $n \in \{1, 2, \ldots\}$, be the subgraph of the $\mathbb{L}^2$ whose vertexes are contained in the plane domain $[-\beta n, \beta n] \times [0, n]$. Then there exists $c_1(\beta)$ such that*

$$HM(W_x) \geq \frac{c_1(\beta)}{n^2}$$

in the Dobrushin domain (R_n^β, u, u) for all $x = (x_1, 0)$ and $u = (u_1, n)$ such that $|x_1|, |u_1| \leq \frac{\beta n}{2}$, so that u and x are "in the middle of the rectangle", where W_x is any of the 2 white diamonds adjacent to the black diamond of x that are not in the dual free arc.

Proof. The situation and the technique of the proof are presented in the figures 54 and 55, respectively. First, let us note that in the domain (R_n^β, u, u), the extra layer of white diamonds adjacent tot he wired arc consists of just one diamond; let us abuse the notation and denote this diamond also by u. Let us define a new lattice graph G as the part of the dual lattice contained in $(R_n^\beta, u, u)_{\partial\diamond}$ with the diamond u added; note that G has the shape of a rectangle. Let us note that the above defined random walk $(Y_n^{W_x})_{n\in\mathbb{N}}$, let us denote it from now on by Y, looks like the simple random walk inside the domain $(R_n^\beta, u, u)_{\partial\diamond}$ in the sense that as long as the walker does not arrive at a white diamond (there are three diamonds of this type in G) of G whose neighbour is u, the random walk behaves just as the simple random walk on G. Hence we can use the results presented in the appendix B to study the random walker Y (this is of course the reason we studied those results in the appendix).

Let us study the function

$$D(x, y) = P(\{Y^{(x,y)} \text{ hits } u \text{ before } \partial G \setminus \{u\}\}),$$

where $Y^{(x,y)}$ is a modified version of the random walk Y such that $Y^{(x,y)}$ begins walking at $(x, y) \in V(G)$ (we identify the diamonds with the plane coordinates corresponding to the dual lattice vertexes that correspond to the diamonds) and ∂G is the boundary of G (those vertexes in $V(G)$ having less

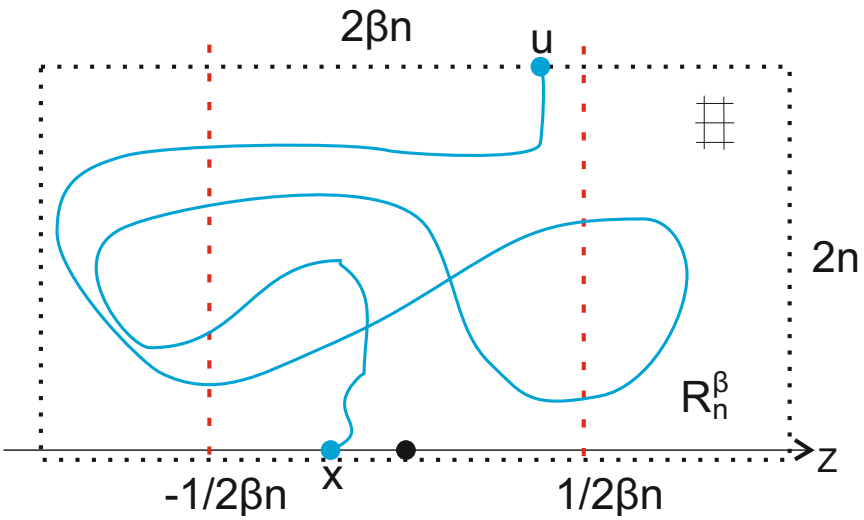


FIGURE 54. The situation of Lemma 7.3. The red dotted lines represent the boundary for the choice of x and u.

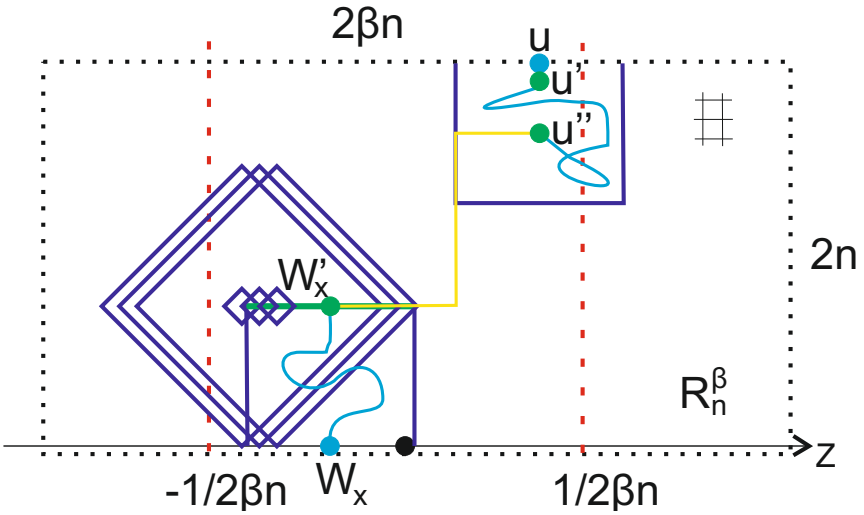


FIGURE 55. The proof of Lemma 7.3. The blue boxes around W_x and the extra layer white diamond adjacent to u are the start and finish rectangles, the green line is the finish line of the start rectangle with the middle point W_x', u' is the point below u, and u'' is the middle point of the finish rectangle, from which we attempt to random walk to u'. The yellow line shows the path which we need to cover with the blue "Harnack balls".

than 4 neighbours in G). Similarly as in the proof of Theorem 7.12, we see that D is a discrete harmonic function for the Laplacian that generates the random walk Y, which is the normal discrete Laplacian (we mean, similar to (7.21)) in the domain G except at diamonds neighbouring u. Let us

now define a "start rectangle" of height $\frac{\beta n}{5}$ and width $\frac{\beta n}{2}$ such that W_x is width-wise in the middle of this rectangle, and the bottom of the rectangle coincides with the bottom of G; let us denote the boundary of this rectangle by ∂S. Let A be the event that the random walker Y, started at W_x, meets ∂S for the first time at the top boundary of the rectangle, and let τ be *stopping time* of Y hitting ∂S, that is, $\tau = \inf\{n \in \mathbb{N} \,|\, Y_n \in \partial S\}$. Now we have that $D(W_x) = E_{W_x}(D(Y_\tau))$, which can be proven in a completely similar manner (by maximum principle of discrete harmonic functions, see appendix Lemma A.1; note that D is discrete harmonic with respect to the normal discrete Laplacian in the start rectangle) as in the proof of appendix Lemma B.4. Similarly as in the above mentioned appendix Lemma B.4, since D is a positive function, we can approximate

$$D(W_x) = E_{W_x}(D(Y_\tau)) \geq E_{W_x}(\chi_{\{Y \in A\}} D(Y_\tau)).$$

Then we use the Harnack inequality of appendix Lemma B.4 for the discrete harmonic function D; note that we use the Harnack inequality for D only in those parts of the domain G where D is discrete harmonic with respect to the normal discrete Laplacian, so no problems in using the appendix Lemma B.4 arise (we can think that we use the Harnack inequality only in such rectangular domains which do not come close to u; these rectangular domains that surround the balls of which the Harnack inequality speaks of are implicitly present at all times we speak of "Harnack balls" below, and as said, D is discrete harmonic in these rectangles with respect to the normal discrete Laplacian as long as these rectangles do not come so close to u that u or some neighbour of u would fall into them). The Harnack inequality says that in a ball like the one in the statement of the Harnack inequality, the values of D vary such that the magnitudes of the values are related by a multiplicative constant. Now we note that we can fit a finite number, depending only on β but not on n (because as n grows bigger, we can make the "Harnack balls" bigger; the only relevant thing is the ration of the rectangular domain and the balls, which depends only of β) of these "Harnack balls" in G such that the "Harnack balls" cover the top boundary of the start rectangle, and that these Harnack balls "stay away" from the area near u where D is no longer discrete harmonic for the normal discrete Laplacian. Then by using Harnack inequality repeatedly, we can show that $H \geq d(\beta) D(W'_x)$ on the top boundary of the start rectangle, where the "constant" $d(\beta) > 0$ (we call $d(\beta)$ a constant because it is independent of n) and W'_x is the point of the top boundary of the start rectangle directly above W_x. Now we apply the appendix Lemma B.1 saying that, starting at the face W_x (which is located one step above the bottom of the start rectangle), the probability of exiting the start rectangle through the top boundary (that is, hitting the boundary of the start rectangle for the first time at the top boundary) is greater than some

constant times $\frac{1}{\frac{\beta n}{5}}$; this leads to a lower bound:

$$\begin{aligned} E_{W_x}(\chi_{\{Y\in A\}}D(Y_\tau)) &\geq E_{W_x}(\chi_{\{Y\in A\}}d(\beta)D(W'_x)) \\ &= d(\beta)D(W'_x)P_{W_x}(\{Y\in A\}) \\ &\geq d(\beta)D(W'_x)\frac{d'(\beta)}{n}, \end{aligned}$$

for some "constant" $d'(\beta) > 0$.

Now let us define another auxiliary domain inside G, the "finish rectangle", as the rectangular domain that has u as its width-wise centre, and whose top boundary corresponds to the top boundary of G, and whose dimensions are the same as the start rectangle's. Let u' be the diamond below u and let u'' be the middle point of the finish rectangle, id est, the point directly below u in the height-wise middle of the rectangle (in the sense of the appendix Lemma B.3). Now let us use Harnack inequality again for D; we note that we can introduce a finite number (not depending on n but only of β) of "Harnack balls" in the domain G such that these Harnack balls span a path from the point W'_x of the "start rectangle" to the point u'' and "stay away from u", meaning that the Harnack balls do not enter the region of G where D is no longer discrete harmonic for the normal discrete Laplacian, see 7.3. Then a repeated use of the Harnack inequality tells us that the magnitudes of the values of the function D in these two points are related by multiplicative constants, also we get $D(W'_x) \geq d''(\beta)D(u'')$. Now using the appendix Lemma B.3 in a rectangle that is otherwise the same as the finish rectangle but that is one step lower than the finish rectangle so that u and all the points of the finish rectangle that are on the same "height" on G as u are left out from the new rectangle so that in the interior of the new rectangle the random walk Y is a simple random walk (note that Y is not simple on the boundary of the new rectangle, but this does not matter when we use the appendix Lemma B.3, because although the Lemma B.3 concerns only simple random walk, the Lemma B.3 also studies only the case where the walk happen in the interior of the rectangle, and the walk is stopped immediately as the boundary is reached, so that is does not matter that Y deviates from simple random walk on the boundary), we can see that the probability of performing a random walk inside this new rectangle and thus inside the finish rectangle (not stepping to the boundary of the finish rectangle) from u'' to u' is bounded below by $\frac{d'''(\beta)}{n}$ for some $d'''(\beta) > 0$. Now we directly see from the definition of D that obviously $D(u') \geq \frac{1}{3+\frac{2}{\sqrt{2}+1}}$. Now since the random walker Y takes all steps independently, we see that $D(u'') \geq \frac{d'''(\beta)}{n} \cdot \frac{1}{3+\frac{2}{\sqrt{2}+1}}$. Now putting all the pieces together, we get

$$D(W_x) \geq \frac{c_1(\beta)}{n^2}$$

for some $c_1(\beta) > 0$. □

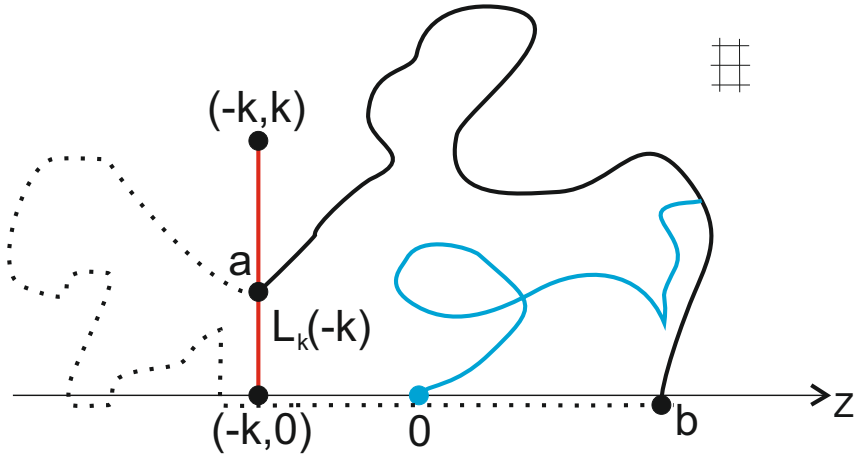


FIGURE 56. The situation of Lemma 7.4.

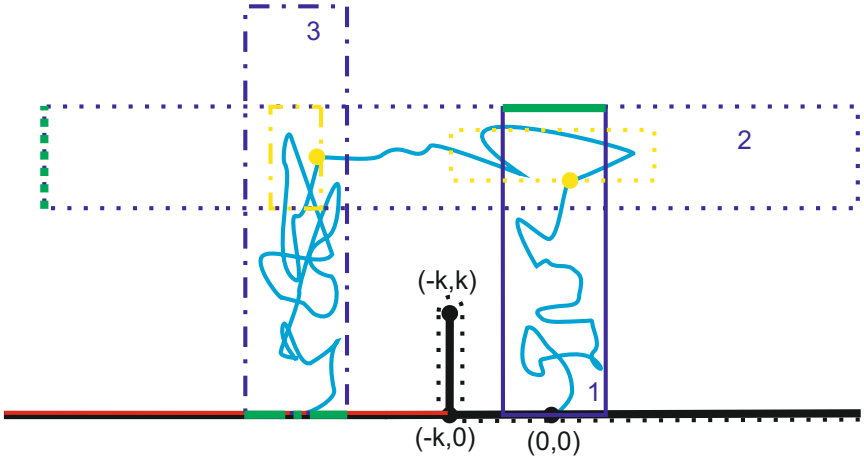


FIGURE 57. The proof of Lemma 7.4. The wired boundary is in red, the free boundary consists of the slit $L_k(-k)$ and the x-axis to the right side of the slit, the rectangles are in blue with different types of borderline, the starting areas are in yellow and finish lines are in green.

Lemma 7.4. *For some constant $c_2 > 0$, for any Dobrushin $\mathbb{H}$-domain (D, a, b) in which the segment $L_k(-k)$ disconnects the origin from the part of the dual free arc that is above the horizontal axis, it holds that*

$$HM(W_0) \geq \frac{c_2}{k}.$$

Proof. Now let us denote the connected component of $D \setminus L_k(-k)$ containing the origin as D_0. Let us put free boundary conditions along $L_k(-k)$ in D_0. Now, since the segment separated the (dual) free arc of D that was above the horizontal axis from the origin, we see that in the new domain D_0 the harmonic measure $\mathrm{HM}(W_0)$ of the wired arc is smaller than the harmonic measure of the wired arc in D, because the number of paths (by a path we

mean the trajectory of the random walker, id est, we do not only mean the collection of diamonds that the random walker visits, but we demand that the order and multiplicity of visitations be also taken into consideration, so that two trajectories in which the random walker visits in total the same diamonds, but at different phases of the walk, are different trajectories) that are favourable for the event of the random walker hitting the wired arc before the dual free arc has decreased (since some paths that were favourable in the original domain are not that any more in D_0, whereas all path that are favourable in D_0 are favourable in the original domain). Note also, that since we are walking on white diamonds and making changes to the (dual) free arc, we do not change the probability weights of the paths (the extra layer next to wired arc remains in place). On the other hand, the harmonic measure of the wired arc in D_0 is larger or at least equal to the harmonic measure of the wired arc in the slit domain $\mathbb{H} \setminus L_k(-k)$ that has wired boundary conditions to the left of the slit and free boundary conditions on the slit and to the right of the slit, since all favourable paths in the new domain are also favourable in the old domain. Note also that probability weights of the paths that in the old domain D_0 "travelled" right next to the extra layer of white diamonds next to the wired arc decrease in the new domain, because our random walk is such that it avoids stepping on the extra layer, so travelling next to the extra layer gives the walk a "boost" in probability weight; in the new domain these walks that followed the wired arc in the old domain no longer necessarily follow the wired arc, so they lose their "boost". See figure 56.

The harmonic measure of the wired arc in the slit domain can be approximated using the results proven in the appendix B. More precisely, we can make a setup like that in figure 57, setting up 3 rectangles R_i, $i = 1, 2, 3$ like in the picture, such that the first rectangle has width and height $3k$, the second has width $8k$ and height k, and the third is of width k and height $4k$, where "width" and "height" refer to figure 57. The rectangles are chosen such that we can ignore boundary effects and just consider the random walker Y, which now starts at W_0, to be a simple random walker except at the very end when Y steps to the extra layer next to the wired arc; note that in the part of the domain right of $L_k(-k)$, Y is exactly a simple random walker. Now, by appendix Lemma B.1, we know that the probability that a random walker, starting at the origin in the middle of rectangle 1, leaves the rectangle through the green top line of the rectangle, call this event $A(1)$, is of the order $\frac{1}{2k}$. But, when travelling towards the top of rectangle 1, the random walker must arrive at the yellow "starting area" of rectangle 2. Then, by appendix Lemma B.2, the probability that the random walker meets the boundary of the second rectangle at the green line of the second rectangle, call this event $A(2)$, is bounded above zero by a positive constant depending only of the aspect ratio of the second rectangle and the relative size of the starting area to the whole second rectangle, both of these factors being constant. But again, on his way to the green line of the second rectangle,

the random walker must arrive at the yellow "starting area" of the third rectangle, and then we can again use the result of appendix Lemma B.2 that the probability that the random walker leaves the third rectangle through the green "finish line" of the third rectangle, call this event $A(3)$ respectively, is bounded above zero uniformly in the size of the third rectangle, that is, uniformly in k. At last, when the random walker comes near (that is, next to) the extra layer next to the wired arc, the last step to the extra layer costs a multiplicative positive constant, $\frac{1}{3+\frac{2}{\sqrt{2}+1}}$ to be exact. Now we can deduce:

$$\mathrm{HM}(W_0) \geq P(\{Y \in A(1)\}) \cdot P(A(2)) \cdot P(A(3)) \cdot \frac{1}{3+\frac{2}{\sqrt{2}+1}} \geq \frac{c_2}{k}$$

for some $c_2 > 0$. □

Lemma 7.5. *There exist constants c_3, $c_4 > 0$ such that for all Dobrushin $\mathbb{H}$-domains (D, a, b),*

$$HM(B_0) \leq c_3 \frac{1}{d},$$

$$HM(B_0) \leq c_4 \frac{n}{k^2},$$

where d is the distance between the origin and the wired arc, and for the second equation, we assume that the segment $L_n(k)$ separates the wired arc from the origin inside D.

Proof. Let us prove the first estimate. For this, we define a Dobrushin domain $(B^{\mathbb{H}}(d), (-d, 0), (d, 0))$ where $B^{\mathbb{H}}(d) = \{x \in \mathbb{Z}^2 \cap \mathbb{H} \mid \delta(0, x) \leq d\}$. Again, due to the number of favourable paths increasing, the harmonic measure of the wired arc in this new domain is greater than in the original domain (D, a, b), see figure 58. Note however that now since we have changed (as opposed to Lemma 7.4) our random walk from walking on the white diamonds to walking on the black diamonds, the probability weights of random walk trajectories do not change as the diamonds on the wired arc are brought to or removed from the vicinity of the path (id est, the changes to the wired arc do not effect the random walk probability weights), but the weights are instead effected by the changes to the dual free arc (or more accurately, by the changes to the extra layer of black diamonds adjacent to the dual free arc). Furthermore, it is the case that this change in the probability weight of a trajectory aspires to negate the effect of increasing number of favourable paths to the harmonic measure of the wired arc when one changes the domain from D to $B^{\mathbb{H}}(d)$, for if there is a favourable path in the domain D that, at some point of its course, follows the dual free arc above the x-axis closely, that is, comes to a diamond B having some number s ($s = 1, 2, 3$, $s = 4$ is obviously impossible) of neighbours in the extra layer of black diamonds adjacent to the dual free arc in the domain $\overline{D_{\partial\diamond}}$ such that these neighbours are above the x-axis (we shall call diamonds like this "upper half plane boundary diamonds" in this proof; note that we do not consider

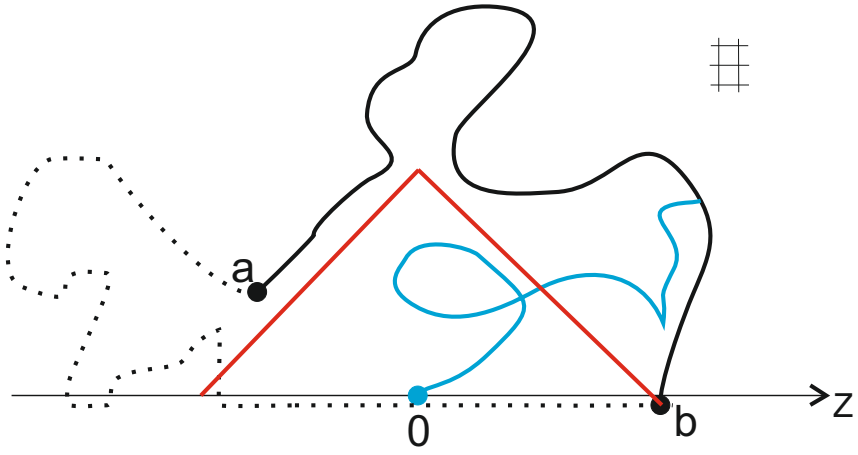


FIGURE 58. The situation in the first part of Lemma 7.5.

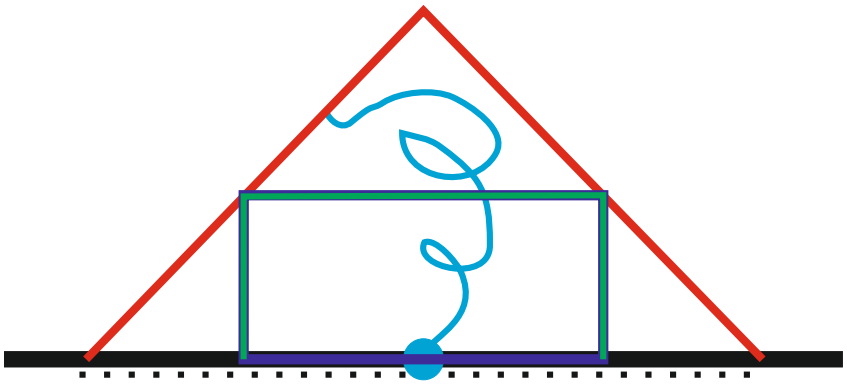


FIGURE 59. The proof of the first part of Lemma 7.5. The wired arc is in red, the free arc is along the x-axis, the auxiliary rectangle is in blue with green "finish line".

the diamonds of the wired arc to be this type of diamonds in this proof, nor are we interested in those black diamonds of $\overline{D_{\partial\diamond}}$ having neighbours in the extra layer of black diamonds such that all these neighbours are below the x-axis - note that if a black diamond does have neighbours in the extra layer of black diamonds both above and below the x-axis, then it is considered an "upper half plane boundary diamond" - because in our new domain $B^{\mathbb{H}}(d)$ these diamonds that have all extra layer neighbours below x-axis remain in the same configuration-status, and the number of neighbours they have on the extra layer of black diamonds in the new domain $\overline{B^{\mathbb{H}}(d)_{\partial\diamond}}$ is the same as the number of these kind of neighbours they had in the old domain $\overline{D_{\partial\diamond}}$ since the free arc in $B^{\mathbb{H}}(d)$ is made to coincide with the x-axis), then the probability weight of this path, consisting of some factors of the form $\frac{1}{(4-s')+s'\frac{2}{\sqrt{2}+1}} \geq \frac{1}{4}$ for $s' = 0, 1, 2, 3$, picks up a factor with $s' = s'_D(B) \neq 0$ when the path travels through this diamond, so that the probability weight

of this path in the domain D is greater than ($\frac{1}{(4-s)+s\frac{2}{\sqrt{2}+1}} > \frac{1}{4}$ for $s = 1, 2, 3$) the same path in the domain $B^{\mathbb{H}}(d)$, because in the domain $B^{\mathbb{H}}(d)$ the path does not travel next to the dual free arc above the x-axis and thus its probability weight picks up a factor with smaller $s' = s'_{B^{\mathbb{H}}(d)}(B) < s'_D(B)$, also the factor picked up by the probability weight is smaller as the factors increase with s' (usually $s' = 0$, also the factor is $\frac{1}{4}$), when it comes to this diamond B or to other upper half plane boundary diamonds introduced above, and the number of factors in the probability weight remains the same in both domains (as the number of steps the random walker takes remains the same). So when we change the domain from D to $B^{\mathbb{H}}(d)$, some favourable paths lose probability mass (but all favourable paths in D remain favourable in $B^{\mathbb{H}}(d)$), but the increase in the number of favourable paths leads to the total probability mass of random walks hitting the wired arc before the extra layer of black diamonds adjacent to the dual free arc (id est, the harmonic measure of the wired arc) to increase. This is what happens, but of course this demands some argumentation to show that these changes to the dual free arc do not counteract the effect of number of favourable trajectories increasing, so that the combined effect is the harmonic measure of the wired arc increasing when one changes the domain from D to $B^{\mathbb{H}}(d)$.

So let us study some random walk trajectory T that has N steps and that is favourable in D. Now it is clear that this particular trajectory has a finite number of steps and thus it steps onto an upper half plane boundary diamond only finitely many times (if the trajectory returns to some upper half plane boundary diamond many times, each time is considered a separate "event of stepping onto upper half plane boundary diamond"). So let B be the first upper half plane boundary diamond the walk meets. Let us assume that the walk does not meet any black diamonds of graph distance greater or equal to d before B, because if it does, then since these black diamonds are in the wired arc of $B^{\mathbb{H}}(d)$, we can stop the walk at the first time it meets one diamond of this type, and thus we are able to increase the probability of hitting the wired arc by restricting the path to this initial segment (obviously this new initial segment includes in its probability mass all the trajectories T that have the same initial segment and are favourable in D, and indeed the initial segment even overestimates the probability mass of these kind of trajectories by it own probability weight). Let us assume that B has $s = 1, 2, 3$ neighbours in the extra layer of black diamonds above x-axis, and let us assume that none of these neighbours of B are such that they become part of the wired arc or fall outside the domain when we make the transition from domain D to $B^{\mathbb{H}}(d)$. Now the probability weight of T in D is of the form (note that we start on the origin which is on the free arc)

$$P_D(T) = \frac{1}{3 + \frac{2}{\sqrt{2}+1}} \cdot \left(\frac{1}{4}\right)^{K-1} \cdot \frac{1}{(4 - s') + s'\frac{2}{\sqrt{2}+1}} \cdot P_D(T')$$

where K is the number of steps the walker takes to reach B, s' is the number of neighbours B has on the extra layer of black diamonds - note that it may be the case that $s' = s + 1$ (but not $s' = s + r$ for $r \geq 2$ because a black diamond inside the domain can have only one neighbouring black diamond that is below x-axis on the extra layer of black diamonds since we are dealing with Dobrushin $\mathbb{H}$-domains) but in any case $s' \in \{1, 2, 3\}$ - and T' is the trajectory followed by the walker after stepping out of B (note that stepping into B does not result in a special factor, but stepping out does). It is also possible that B coincides with the origin, so that $K = 0$; in this case it is understood that the probability $P(T)$ does not have the factors before $\frac{1}{(4-s')+s'\frac{2}{\sqrt{2}+1}}$ present at all. Now let us study random walk trajectories T_i^j, $i \in \mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$, $j = 1, \ldots, s$ in $B^{\mathbb{H}}(d)$ that are as follows: the first K steps of each T_i^j are the same as for T, and also the last segment of each T_i^j follows the trajectory T' as does T. The difference between T_i^j as T is what happens when the walk comes to B. Namely, let us enumerate the neighbours of B in the extra layer of black diamonds above the x-axis, with the index $j = 1, \ldots, s$. Then, the walk T_i^j, when arriving at B, steps i times (note $i = 0$ is a possibility) to the neighbour j of B and back to B; this manoeuvre of one back-and-forth step costs a factor $\frac{1}{(4-s'')+s''\frac{2}{\sqrt{2}+1}} \cdot \frac{1}{(4-s''')+s'''\frac{2}{\sqrt{2}+1}} \geq \frac{1}{16}$, where s'' is the number of extra layer black diamond neighbours of B in $B^{\mathbb{H}}(d)$ (note $s'' = s' - s$) and s''' is the number of extra layer black diamond neighbours of the j-neighbour of B in $B^{\mathbb{H}}(d)$; note that because the free arc coincides with the x-axis in $B^{\mathbb{H}}(d)$, $s'', s''' \in \{0, 1\}$ since both B and B's j-neighbour are above or at the x-axis. Note that all the trajectories T_i^j are such that they are new trajectories, id est, trajectories that are favourable in $B^{\mathbb{H}}(d)$ but not in D. After this back-and-forth pacing, the walk T_i^j continues to follow the trajectory T', as said above. The probability of the walk T_i^j in $B^{\mathbb{H}}(d)$ satisfies

$$P_{B^{\mathbb{H}}(d)}(T_i^j) \geq \frac{1}{3+\frac{2}{\sqrt{2}+1}} \cdot \left(\frac{1}{4}\right)^{K-1} \cdot \left(\frac{1}{16}\right)^{i} \cdot \frac{1}{(4-(s'-s))+(s'-s)\frac{2}{\sqrt{2}+1}} \cdot P_{B^{\mathbb{H}}(d)}(T').$$

Then we just sum over the probabilities $P_{B^{\mathbb{H}}(d)}(T_i^j)$ to find the probability that one of the random walks T_i^j takes place; we find that

$$
\begin{aligned}
&P_{B^{\mathbb{H}}(d)}(\text{One of the random walk trajectories } T_i^j \text{ manifests}) \\
&\geq \sum_{j=1}^{s}\sum_{i=0}^{\infty} \frac{1}{3+\frac{2}{\sqrt{2}+1}} \cdot \left(\frac{1}{4}\right)^{K-1} \cdot \left(\frac{1}{16}\right)^{i} \cdot \frac{1}{(4-(s'-s))+(s'-s)\frac{2}{\sqrt{2}+1}} \cdot P_{B^{\mathbb{H}}(d)}(T') \\
&= s\frac{1}{3+\frac{2}{\sqrt{2}+1}} \cdot + \left(\frac{1}{4}\right)^{K-1} \cdot \frac{1}{(4-(s'-s))+(s'-s)\frac{2}{\sqrt{2}+1}} \cdot P_{B^{\mathbb{H}}(d)}(T') \sum_{i=0}^{\infty}\left(\frac{1}{16}\right)^{i} \\
&= s\frac{16}{15}\frac{1}{3+\frac{2}{\sqrt{2}+1}} \cdot \left(\frac{1}{4}\right)^{K-1} \cdot \frac{1}{(4-(s'-s))+(s'-s)\frac{2}{\sqrt{2}+1}} \cdot P_{B^{\mathbb{H}}(d)}(T').
\end{aligned}
$$

Now the important thing is that

$$
\begin{aligned}
&s\frac{16}{15}\frac{1}{3+\frac{2}{\sqrt{2}+1}} \cdot \left(\frac{1}{4}\right)^{K-1} \cdot \frac{1}{(4-(s'-s))+(s'-s)\frac{2}{\sqrt{2}+1}} \\
&\geq \frac{1}{3+\frac{2}{\sqrt{2}+1}} \cdot \left(\frac{1}{4}\right)^{K-1} \cdot \frac{1}{(4-s')+s'\frac{2}{\sqrt{2}+1}}
\end{aligned}
$$

because

$$
s\frac{16}{15}\frac{1}{(4-(s'-s)+)+(s'-s)\frac{2}{\sqrt{2}+1}} \geq \frac{1}{(4-s')+s'\frac{2}{\sqrt{2}+1}}
$$

for all $s, s' = 1, 2, 3$ where $s' \in \{s, s+1\}$. Then we can just iterate this result with the trajectory T', which now is such that it visits the upper half plane boundary diamonds one time less than T, namely we can construct similar kind of favourable trajectories $(T')_i^j$ and show that their total probability in $B^{\mathbb{H}}(d)$ exceeds that of T' in D. Technically we of course have to combine our trajectories T_i^j and $(T')_{i'}^{j'}$ and those trajectories we would find when we continue the iteration (until we are left with a path that does not meet any upper half plane boundary diamond before hitting the wired arc) such that any combination can be fused into one trajectory from the origin to the wired arc. This of course can be done, and our conclusion above about the total probability of the newly formed paths in $B^{\mathbb{H}}(d)$ being greater than the probability of the path T in domain D still holds precisely as presented above since the random walk step are independent. So indeed we have shown that the increase in the number of favourable paths does outweigh the decrease of the probability weights of (certain) paths.

However, there are still some details that we left unchecked, namely we assumed above that B's upper half plane boundary diamond neighbours were such that none of them became a part of the wired arc nor was left outside the domain as the transition from domain D to domain $B^{\mathbb{H}}(d)$ was made. Let us now look into this detail for a little while in order to provide a complete proof.

So first, assume that some of the above mentioned neighbours of B fall outside the domain in the domain change. What this means is that these neighbours are not in the origin-centred d-radius ball (restricted to the upper half plane), id est, there is no (primal) lattice path of length less or equal to d to these neighbours (note that we are dealing with black diamonds, so they correspond to primal lattice vertexes). Then there either is a d-length path from the origin to B, in which case B is part of the wired arc and thus the random walk path becomes a favourable path already when it hits B and we can discard the rest of the path and increase the probability weight of the path while doing this discarding (so that the probability weight again increases as we move from D to $B^{\mathbb{H}}(d)$), or there is no lattice path of length less or equal to d from the origin to B, so that B itself falls of the domain. But then the path must have crossed the wired arc of the new domain $B^{\mathbb{H}}(d)$ on its way to B, and thus we can restrict ourselves to some initial segment of the path, because this initial segment already must be a favourable path in the new domain, and thus again (by "clipping" the excess off the path) we can increase the probability weight, so that again the total probability can be seen to increase as we change from D to $B^{\mathbb{H}}(d)$.

So then let us study the case that one or more of the neighbours of B that are in the extra layer of black diamonds in D fall into the wired arc in the new domain. Then let T be the trajectory we studied above. Now if the random walker, while travelling according to T, must take two steps after hitting B, then since the probability for two steps is at most $\frac{1}{(4-r)+r\frac{2}{\sqrt{2}+1}} \cdot \frac{1}{(4-r')+r'\frac{2}{\sqrt{2}+1}} \leq \frac{1}{4}$ for some $r, r' \in \{0, 1, 2, 3\}$, we see that the probability that a random walker that follows trajectory T to B and then steps to some of the neighbours of B that became part of the wired arc in the change of the domain is greater than the probability $P_D(T)$ (note also that this trajectory of coming to B according to T and then stepping to the wired arc is new, id est, it is favourable in $B^{\mathbb{H}}(d)$ but not in D), and thus we again see that the probability is increased as one changes the domain. So, the only case we have not yet looked at is that none of upper half plane boundary diamond neighbours of B do not fall off the domain as the domain is changed, and B has some upper half plane boundary neighbours that become part of the wired arc in the new domain, and that a random walker following T can step to wired arc from B with one step, id est, some of the neighbours of B belong to the wired arc already in D. Let s be the number of upper half plane boundary neighbours of B in D. Now because T must arrive at B through some non-boundary diamond and since T cannot hit the wired arc of D before coming to B, we see that the possibilities now are $s = 1, 2$ (actually there is yet another exceptional case; namely, if B is the origin, and we are just about to begin our random walk T, then T need not arrive at B and the case $s = 3$ is actually a possibility, but this is not a problem because we know that one of the neighbours of B must belong to the wired arc in D, so that we see that in the case $s = 3$ we are dealing with a Dobrushin $\mathbb{H}$-domain

D such that the left, right and bottom neighbours of B belong to the extra layer and the top neighbour of B belongs to wired arc, and this case we can analyse by hand and note that the probability that the random walker starting at origin in D hits the wired arc before the extra layer is $\frac{1}{1+3\cdot\frac{2}{\sqrt{2}+1}}$, whereas in the domain $B^{\mathbb{H}}(1)$, which is the corresponding half-ball domain, the probability is $\frac{3}{3+\frac{2}{\sqrt{2}+1}}$, so that the harmonic measure increases). Now let any of the neighbours of B at the wired arc of D be C, then by definition of d, $\delta(0,C) \geq d$ but since no neighbour falls out of $B^{\mathbb{H}}(d)$, $\delta(0,C) \leq d$ where δ is the graph metric. So C is also in the wired arc of $B^{\mathbb{H}}(d)$. Then we note that is $s = 1$, the it is necessarily this neighbour of B that becomes part of the wired arc in $B^{\mathbb{H}}(d)$. This means that in $B^{\mathbb{H}}(d)$, B has at least two neighbours on the wired arc, and one of them is new. Then we see that we get a new path, id est, a path that is favourable in $B^{\mathbb{H}}(d)$ but not in D by stepping to this new wired arc neighbour from B. Now the probabilities of this new path T'' and the path T in $B^{\mathbb{H}}(d)$ sum up to

$$
\begin{aligned}
&P_{B^{\mathbb{H}}(d)}(T'') + P_{B^{\mathbb{H}}(d)}(T) \\
&= \frac{(4-s') + s'\frac{2}{\sqrt{2}+1}}{(4-(s'-s)) + (s'-s)\frac{2}{\sqrt{2}+1}} P_D(T) + \frac{(4-s') + s'\frac{2}{\sqrt{2}+1}}{(4-(s'-s)) + (s'-s)\frac{2}{\sqrt{2}+1}} P_D(T) \\
&= 2\frac{4-s'+s'\frac{2}{\sqrt{2}+1}}{4+s-s'+(s'-s)\frac{2}{\sqrt{2}+1}} P_D(T) \\
&> P_D(T)
\end{aligned}
$$

where s' is the number of extra layer neighbours of B, note $s = 1$ and $s' \in \{1,2\}$. This yet again shows that the total probability increases in the change of domain. Now the final case is $s = 2$. We can use a similar argument as above; in $B^{\mathbb{H}}(d)$, B has at least one new neighbour that is in the wired arc, and we can form the paths T'' and T as above and make the same conclusion about the sum of the probabilities; note that now $s = 2$ and $s' = 2$. So finally we have shown that in any case, the total probability of hitting the wired arc before the extra layer of black diamonds increases as one changes the domain from D to $B^{\mathbb{H}}(d)$.

Now we define an additional domain of a rectangle of height $\frac{1}{2}d$ and width $2 \cdot \frac{1}{2}d = d$ placed on the x-axis as in figure 59. Now we see that the harmonic measure of the wired arc (the probability of random walker, starting at the origin hitting the wired arc before the extra layer along the dual free arc) in the triangular domain is smaller than the probability that the random walker X that starts at the origin, enters the additional rectangle by a step upward (probability $\frac{1}{3+\frac{2}{\sqrt{2}+1}}$), and then hits the boundary of the rectangle for the first time at the right, top or left side of the boundary of the rectangle. Furthermore, since the random walk inside the rectangle is a simple random walk, and the walk only deviates from the simple random walk in the bottom side of the rectangle (and on the lower right and left corners

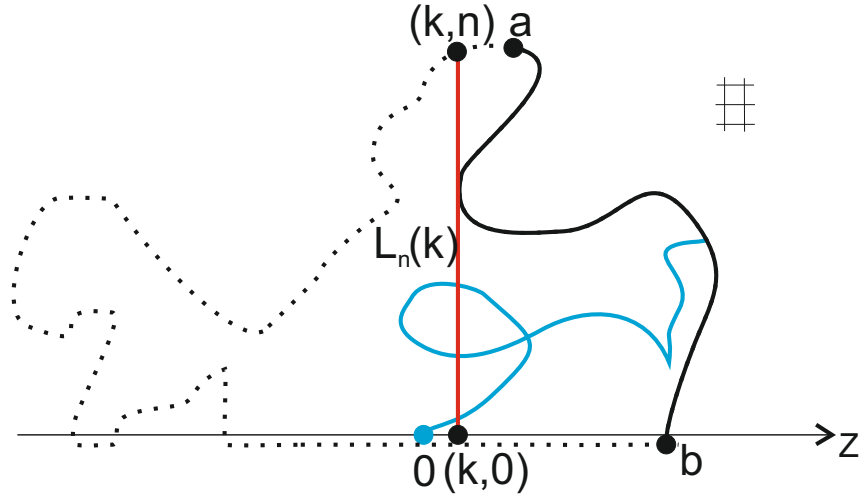


FIGURE 60. The situation in the second part of Lemma 7.5.

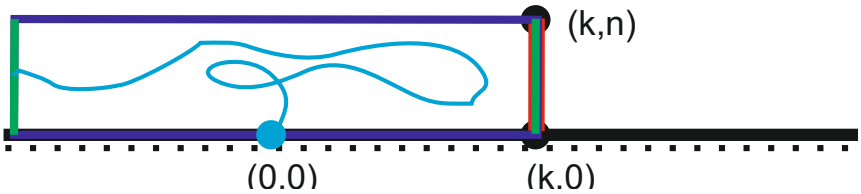


FIGURE 61. The proof of the second part of Lemma 7.5.

of the rectangle), where we have conditioned that the random walk does not go (note that because of the structure of the square lattice, the random walk cannot go to the corners of the rectangle without hitting some part of the boundary first), we can approximate this harmonic measure by methods of the appendix B, namely by studying the proof of appendix Lemma B.1 and modifying the results found there into our purposes here. Let us denote, as in the appendix B, the right, top, left and bottom boundaries of the rectangle $\mathfrak{R}, \mathfrak{T}, \mathfrak{L}$ and $\mathfrak{B}$, respectively, and let us assume that the simple random walker X' we are interested in starts its walk at one lattice step above the bottom, in the width-wise middle site of the rectangle. We see that by the result of the proof of appendix Lemma B.1:

$$
\begin{aligned}
&P(\{X' \text{ hits } \mathfrak{R} \cup \mathfrak{T} \cup \mathfrak{L} \text{ before } \mathfrak{B}\}) \\
&= P(\{X' \text{ hits } \mathfrak{T} \text{ before } \mathfrak{R} \cup \mathfrak{L} \cup \mathfrak{B}\}) + P(\{X' \text{ hits } \mathfrak{R} \cup \mathfrak{L} \text{ before } \mathfrak{T} \cup \mathfrak{B}\}) \\
&\leq \frac{1}{\frac{1}{2}d - 1} + \frac{4}{2^2 \cdot \left(\frac{1}{2}d - 1\right)}.
\end{aligned}
$$

Thus we see that a $c_3 > 0$ that is as claimed exists.

Now we prove the second estimate. Since $L_n(k)$ disconnects the origin from the wired arc, the harmonic measure of the wired arc is smaller than the harmonic measure of $L_n(k)$ (the probability that the random walker comes to some vertex of $L_n(k)$ before hitting the extra layer adjacent to the

dual free arc) in D (for all random walk trajectories leading to the wired arc must pass through $L_n(k)$ but not vice versa), and this harmonic measure is smaller than the harmonic measure of the segment in the domain $(\mathbb{H} \setminus L_n(k), (k, 0), (k, n))$ that has wired boundary conditions in the left side (right side if $k < 0$) of the slit $L_n(k)$ and free elsewhere, due to increase in the number of favourable paths (again the probability weight of some favourable paths decreases as one moves from the domain D to the domain $\mathbb{H} \setminus L_n(k)$ because the dual free arc, or more precisely, the extra layer of black diamonds adjacent to the dual free arc is moved away from the path, but this can dealt with as above to show that still the total harmonic measure is increased).

Now we can approximate the harmonic measure as follows: we study the case that the random walker steps into the rectangle of figure 61, and then exits the rectangle though the right side, ending up at the wired arc (note that the rectangle is positioned on the x-axis, and the inside the rectangle the random walk is the simple random walk; deviations from the simple random walk happen only at the bottom of the rectangle, where the random walk never goes since we condition the random walk to exit the rectangle through the right side). Now the possibility that the random walker takes the first step towards the inside of the rectangle is $\frac{1}{3+\frac{2}{\sqrt{2}+1}}$. Then the random walker behaves as simple random walker inside the rectangle. We can see that, due to the symmetry between right and left for the simple random walk, that the probability of leaving through the left side is half the probability that a random walker, starting at one step above bottom and moving inside the rectangle of 61, hits the right or left side before the top or bottom side. Then we get, directly from the proof of appendix Lemma B.1:

$$\mathrm{HM}(B_0) \leq \frac{1}{3+\frac{2}{\sqrt{2}+1}} \cdot \frac{4}{\left(\frac{k}{n}\right)^2 n} = \frac{4}{3+\frac{2}{\sqrt{2}+1}} \frac{n}{k^2}$$

so $c_4 = \frac{4}{3+\frac{2}{\sqrt{2}+1}}$ is a possible choice. Note that the wired arc does not cause the random walker to deviate from the simple random walker in any way. □

7.4.6. *Proof of box-crossing property.* Now we have done enough work with the harmonic measure estimates and we are ready to proceed towards the actual theorem of box-crossing property. A few more lemmas are needed:

Lemma 7.6. *Let (D, a, b) be a Dobrushin domain, and let x be a site (diamond) in the free arc of D. Let B_x be the black diamond corresponding to x, and let W_x be an adjacent white diamond not on the dual free arc of $D_{\partial\diamond}$ (supposing such a diamond exist, for if x is part of a "spike" of the domain D, then the dual free arc can circle x; in this case we agree that W_x is one of the neighbouring white diamonds that is on the dual free arc and thus*

$HM(W_x) = 0$*, a trivial lower bound). We show:*

$$\sqrt{HM(W_x)} \leq P_{(D,a,b)}(x \leftrightsquigarrow \text{ wired arc}) \leq \sqrt{HM(B_x)}.$$

Proof. Because x is on the free arc of D, there exist a white diamond of $D_{\partial\diamond}$ adjacent to B_x; let e be the edge of $D_\diamond$ between these diamonds. Now the winding $W_\gamma(e_a, e)$ of the exploration path γ at e is a constant, since e is in the free arc, and the winding depends only on the direction of e. This implies (by the definition of F)

$$P_{(D,a,b)}(e \in \gamma) = |F(e)|.$$

Also, we see that $e \in \gamma$ if and only if x is connected to the wired arc, so $|F(e)| = P_{(D,a,b)}(x \leftrightsquigarrow \text{ wired arc})$; Theorem 7.12 implies the claim. □

Let us now denote $\partial_+ R_n^\beta = [-\beta n, \beta n] \times \{2n\}$ the *top side* of the rectangle R_n^β and $\partial_- R_n^\beta = [-\beta n, \beta n] \times \{0\}$ the *bottom side*.

Lemma 7.7. *For $\beta > 0$, there exist a constant $c_5(\beta) > 0$ such that for all $n \geq 1$*

$$P^0_{R_n^\beta}(x \leftrightsquigarrow u) \geq \frac{c_5(\beta)}{n}$$

for all $x = (x_1, 0) \in \partial_- R_n^\beta$ and $u = (u_1, 2n) \in \partial_+ R_n^\beta$ such that $|x_1|, |u_1| \leq \frac{\beta n}{2}$.

Proof. The Lemmas 7.6 and 7.3 will prove the claim, since

$$P^0_{R_n^\beta}(x \leftrightsquigarrow u) = P_{(R_n^\beta, u, u)}(x \leftrightsquigarrow \text{ wired arc}).$$

Note that in the Dobrushin-domain (R_n^β, u, u) the wired arc consist of only one vertex and the boundary is otherwise free. □

Lemma 7.8. *There exist a constant $c_6 > 0$ such that for any rectangle R_n^β and any $x \in \partial_- R_n^\beta$, for all $k \geq 1$*

$$P_{(R_n^\beta, a_n, b_n)}(B_k^{\mathbb{H}}(x) \cap R_n^\beta \leftrightsquigarrow \text{ wired arc}) \leq c_6 \sqrt{\frac{k}{n}}, \tag{7.24}$$

where a_n and b_n denote the top-left and top-right corners of the rectangle R_n^β, and $B_k^{\mathbb{H}}(x) = \{y \in \mathbb{Z}^2 \cap \mathbb{H} \mid \delta(x, y) \leq k\}$; δ is the graph metric.

Proof. Consider $n, k, \beta > 0$. Now we note that the case $k \geq n$ is trivial, since we can choose $c_6 = 1$. So assume $k \leq n$ (we can take the case $k \geq n$ into consideration by setting $c_6 = \max\{1, c_6'\}$ where c_6' is such that (7.24) holds for $k \leq n$). Now we note that if we increase β, we are widening the rectangle R_n^β and therefore increasing the possibility that $B_k^{\mathbb{H}}(x) \cap R_n^\beta$ is connected to the wired arc.

This is of course quite intuitive, but let us present a proof for this fact. Let us denote by C the event that there exists two dual open vertical lines in the dual domain $(R_n^{\beta+\epsilon})_d$ of $R_n^{\beta+\epsilon}$, $0 < \epsilon \in \mathbb{N}$, such that these dual lines

are at the boundary of the domain $R_n^\beta \subset R_n^{\beta+\epsilon}$, id est, the dual lines have the x-coordinate equal to $\pm(\beta n + \frac{1}{2})$. Now we note

$$\begin{aligned}&P_{(R_n^\beta, a_n, b_n)}(B_k^{\mathbb{H}}(x) \cap R_n^\beta \leftrightsquigarrow \text{ wired arc})\\&= P_{(R_n^{\beta+\epsilon}, a_n-(\epsilon n,0), b_n+(\epsilon n,0))}(B_k^{\mathbb{H}}(x) \cap R_n^\beta \leftrightsquigarrow \text{ wired arc} \,|\, C)\end{aligned}$$

by Domain Markov property (the event C prohibits all connections from $R_n^{\beta+\epsilon} \setminus R_n^\beta$ to being inherited to R_n^β) where the upper corners of the new domain $R_n^{\beta+\epsilon}$ are $a_n - (\epsilon n, 0)$, $b_n + (\epsilon n, 0)$. Hence

$$\begin{aligned}&P_{(R_n^{\beta+\epsilon}, a_n-(\epsilon n,0), b_n+(\epsilon n,0))}(\{B_k^{\mathbb{H}}(x) \cap R_n^\beta \leftrightsquigarrow \text{ wired arc}\} \cap C)\\&= P_{(R_n^\beta, a_n, b_n)}(B_k^{\mathbb{H}}(x) \cap R_n^\beta \leftrightsquigarrow \text{ wired arc}) P_{(R_n^{\beta+\epsilon}, a_n-(\epsilon n,0), b_n+(\epsilon n,0))}(C).\end{aligned}$$

Now since C is a decreasing (C is an increasing event in the dual model, hence it is decreasing in the primal model) and $\{B_k^{\mathbb{H}}(x) \cap R_n^\beta \leftrightsquigarrow \text{ wired arc}\}$ is an increasing event, by FKG inequality

$$\begin{aligned}&P_{(R_n^\beta, a_n, b_n)}(B_k^{\mathbb{H}}(x) \cap R_n^\beta \leftrightsquigarrow \text{ wired arc}) P_{(R_n^{\beta+\epsilon}, a_n-(\epsilon n,0), b_n+(\epsilon n,0))}(C)\\&\leq P_{(R_n^{\beta+\epsilon}, a_n-(\epsilon n,0), b_n+(\epsilon n,0))}(B_k^{\mathbb{H}}(x) \cap R_n^\beta \leftrightsquigarrow \text{ wired arc}) P_{(R_n^{\beta+\epsilon}, a_n-(\epsilon n,0), b_n+(\epsilon n,0))}(C).\end{aligned}$$

This then finally gives us

$$\begin{aligned}&P_{(R_n^\beta, a_n, b_n)}(B_k^{\mathbb{H}}(x) \cap R_n^\beta \leftrightsquigarrow \text{ wired arc})\\&\leq P_{(R_n^{\beta+\epsilon}, a_n-(\epsilon n,0), b_n+(\epsilon n,0))}(B_k^{\mathbb{H}}(x) \cap R_n^\beta \leftrightsquigarrow \text{ wired arc})\\&\leq P_{(R_n^{\beta+\epsilon}, a_n-(\epsilon n,0), b_n+(\epsilon n,0))}(B_k^{\mathbb{H}}(x) \cap R_n^{\beta+\epsilon} \leftrightsquigarrow \text{ wired arc}).\end{aligned}$$

So let us have $\beta' = \beta + 2$, and we study from here onwards the probability

$$P_{(R_n^{\beta'}, a_n', b_n')}(B_k^{\mathbb{H}}(x) \cap R_n^{\beta'} \leftrightsquigarrow \text{ wired arc}) \geq P_{(R_n^\beta, a_n, b_n)}(B_k^{\mathbb{H}}(x) \leftrightsquigarrow \text{ wired arc}),$$

where we denote the upper right and left corners of $R_n^{\beta'}$ by a_n' and b_n', respectively. Now since $x \in \partial_- R_n^\beta$ and $k \leq n$, we can see that the half-ball $B_k^{\mathbb{H}}(x) \subset \mathbb{H}$ is contained in the rectangle $R_n^{\beta'}$ and furthermore, the right-most site of $B_k^{\mathbb{H}}(x)$ is at least at a distance n from the wired arc of $R_n^{\beta'}$.

Now, by the monotonicity of the boundary conditions,

$$P_{(R_n^{\beta'}, a_n', b_n')}(B_k^{\mathbb{H}}(x) \leftrightsquigarrow \text{ wired arc}) \leq P_{(R_n^{\beta'}, c_n', d_n')}(B_k^{\mathbb{H}}(x) \leftrightsquigarrow \text{ wired arc}),$$

where c_n' and d_n' are the left and right lower corners of the rectangle $R_n^{\beta'}$ (we have increased the size of the wired arc from consisting of the top side to consisting of the top side and vertical sides).

Let us denote by $T(\omega)$ (we write the dependence on ω explicitly to avoid confusion further down the road) the hitting time of the exploration path $\gamma(\omega)$ (the exploration path observed in configuration ω) parametrized by the number of steps to the set of medial lattice edges that are adjacent to the site-diamonds of $B_k^{\mathbb{H}}(x)$; naturally $T(\omega) = \infty$ if the exploration path does not reach this set; so we see $T(\omega) < \infty$ if and only if $B_k^{\mathbb{H}}(x)$ is connected to the wired arc. Let now z be the right-most site of the $B_k^{\mathbb{H}}(x)$; consider the event $\{z \leftrightsquigarrow \text{ wired arc}\}$. By conditioning on the event $\{B_k^{\mathbb{H}}(x) \leftrightsquigarrow \text{ wired arc}\}$

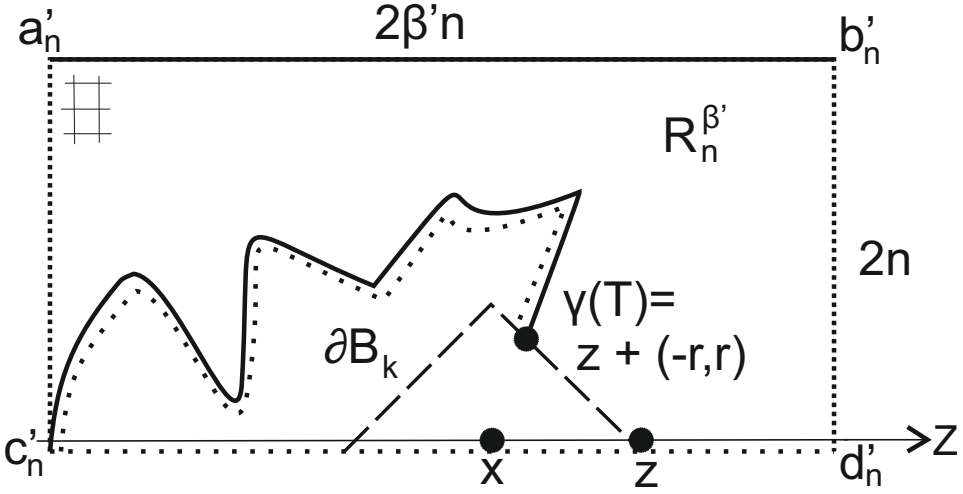


FIGURE 62. The domain $(R_n^{\beta'}, c_n', d_n')$ with exploration path up to time T.

and by the exploration path up to time T (denote this condition by $\gamma[0,T]$; note that now we mean that one first picks a deterministic prospective exploration path up to time T, that is, a prospective initial segment of an exploration path such that this initial segment ends at hitting a medial edge adjacent to a vertex in $B_k^{\mathbb{H}}(x)$, from the set of all possible exploration paths and then conditions on it, and therefore we do not denote the path or the stopping time as a function of ω; note that since our domain is finite and the exploration path is non-self-intersecting, there are only a finite number of possible exploration paths) we get (note that obviously we have that $\{z \leftrightsquigarrow \text{ wired arc}\} \subset \{B_k^{\mathbb{H}}(x) \leftrightsquigarrow \text{ wired arc}\}$)

$$
\begin{aligned}
&P_{(R_n^{\beta'}, c_n', d_n')}(z \leftrightsquigarrow \text{ wired arc}) \\
&= \sum_{\substack{\gamma[0,T] \\ T<\infty}} P_{(R_n^{\beta'}, c_n', d_n')}(\{z \leftrightsquigarrow \text{ wired arc}\} \cap \{\gamma[0,T]\}) \\
&= \sum_{\substack{\gamma[0,T] \\ T<\infty}} P_{(R_n^{\beta'}, c_n', d_n')}(z \leftrightsquigarrow \text{ wired arc} \,|\, \gamma[0,T]) P_{(R_n^{\beta'}, c_n', d_n')}(\gamma[0,T]) \\
&= \sum_{\substack{\gamma[0,T] \\ T<\infty}} P_{(R_n^{\beta'}, c_n', d_n')}(z \leftrightsquigarrow \text{ wired arc} \,|\, \gamma[0,T]) \sum_{\substack{\omega \in \Omega_{R_n^{\beta'}} \\ \omega \text{ has } \gamma[0,T]}} P_{(R_n^{\beta'}, c_n', d_n')}(\omega) \\
&= \sum_{\omega \in \Omega_{R_n^{\beta'}}} \chi_{T<\infty}(\omega) P_{(R_n^{\beta'}, c_n', d_n')}(z \leftrightsquigarrow \text{ wired arc} \,|\, \gamma[0,T])(\omega) P_{(R_n^{\beta'}, c_n', d_n')}(\omega) \\
&= E_{(R_n^{\beta'}, c_n', d_n')}(\chi_{T<\infty}(\omega) P_{(R_n^{\beta'}, c_n', d_n')}(z \leftrightsquigarrow \text{ wired arc} \,|\, \gamma[0,T])(\omega)) \\
&= E_{(R_n^{\beta'}, c_n', d_n')}(\chi_{T<\infty}(\omega) P_{(R_n^{\beta'} \setminus \gamma[0,T[, \gamma(T), d_n')}(z \leftrightsquigarrow \text{ wired arc})(\omega)),
\end{aligned}
$$

where we have used the Domain Markov property as well as the notion that it is enough for z to be connected to the wired arc in the original domain that z is connected to the wired arc in the new slit domain, since the side of the exploration path that is made wired in the slit domain must be connected to the wired arc in the original domain, and otherwise there are no changes to the wired arcs. Note also that when we examine the event $\{z \leftrightsquigarrow \text{ wired arc}\}$, we have always $T(\omega) < \infty$ for configurations ω that are in the event, and so $\gamma[0,T](\omega)$ is a well-defined medial-lattice path; furthermore, since $R_n^{\beta'}$ is a finite lattice domain, then there are only a finite number of possible exploration paths $\gamma[0,T]$ (the exploration path cannot intersect itself) and in every configuration in the event $\{z \leftrightsquigarrow \text{ wired arc}\}$ there is an unique exploration path, so we can partition the event into separate pieces by considering the exploration paths of the configurations. Note also that we define the functions

$$\begin{aligned}&P_{(R_n^{\beta'},c_n',d_n')}(z \leftrightsquigarrow \text{ wired arc} \,|\, \gamma[0,T])(\omega)\\ &= P_{(R_n^{\beta'},c_n',d_n')}(z \leftrightsquigarrow \text{ wired arc} \,|\, \gamma[0,T](\omega))\end{aligned}$$

and

$$\begin{aligned}&P_{(R_n^{\beta'}\setminus\gamma[0,T[,\gamma(T),d_n')}(z \leftrightsquigarrow \text{ wired arc})(\omega)\\ &= P_{(R_n^{\beta'}\setminus\gamma[0,T[(\omega),\gamma(T)(\omega),d_n')}(z \leftrightsquigarrow \text{ wired arc}),\end{aligned}$$

which are measurable (because our domain is finite, the σ-algebra is the power set; because of the finiteness of the domain the measurability is clear) real functions, also random variables. We also denote the exploration path observed in configuration ω by $\gamma[0,T](\omega) = \gamma(\omega)[0,T(\omega)]$; note that both the path $\gamma(\omega)$ and the stopping time $T(\omega)$ are functions of ω. We also denote $\gamma(T)(\omega) = \gamma(\omega)(T(\omega))$. Furthermore, because of the finiteness of the domain $R_n^{\beta'}$, the set of configurations is finite, and because of the random cluster measure, each configuration has a strictly positive probability, and therefore the expected value is a sum over the set of configurations.

Now, since z is at least distance n from the wired arc (as we increased the β), Lemmas 7.6 and 7.5 give us

$$P_{(R_n^{\beta'},c_n',d_n')}(z \leftrightsquigarrow \text{ wired arc})(\omega) \leq \frac{c_6}{\sqrt{n}}.$$

Now, by the definition of $T(\omega)$, if $T(\omega) < \infty$, then $\gamma(T)(\omega)$ can be expressed either as $\gamma(T)(\omega) = z + (-r, r)$, where $0 \leq r \leq k$, or in the other case we have $\gamma(T)(\omega) = z + (-r, 2k - r)$ where $k + 1 \leq r \leq 2k$. Let $r > 0$. Now we see that the arc $z + L_r(-r)$ disconnects the part of the free arc of $R_n^{\beta'} \setminus \gamma[0,T[(\omega)$ that is above the horizontal axis from z in the slit-domain $R_n^{\beta'} \setminus \gamma[0,T[(\omega) = R_n^{\beta'} \setminus \gamma(\omega)[0,T(\omega)[$. Now using Lemmas 7.6 and 7.4 we get that

$$P_{(R_n^{\beta'}\setminus\gamma[0,T[,\gamma(T),d_n')}(z \leftrightsquigarrow \text{ wired arc})(\omega) \geq \frac{c_2}{\sqrt{r}} \geq \frac{c_2}{\sqrt{2k}}.$$

The special case $r = 0$, also $\gamma(T) = z$, can be handled as follows:

$$P_{(R_n^{\beta'} \setminus \gamma[0,T[,z,d_n')}(z \leftrightsquigarrow \text{ wired arc})(\omega) = 1 \geq \frac{c_2}{\sqrt{2k}},$$

for $k \geq 1$.

Since the estimate above does not depend on what the path $\gamma[0,T](\omega)$ is, we get

$$\frac{c_2}{\sqrt{2k}} P_{(R_n^{\beta'},c_n',d_n')}(T < \infty) \leq P_{(R_n^{\beta'},c_n',d_n')}(z \leftrightsquigarrow \text{ wired arc}) \leq \frac{c_6}{\sqrt{n}},$$

which gives us the claim (as we return to the original β). □

Lemma 7.9. *There exist a constant $c_7 > 0$ such that for any rectangle R_n^β and any 2 points $x, y \in \partial_- R_n^\beta$, $x \neq y$:*

$$P_{(R_n^\beta,a_n,b_n)}(x, y \leftrightsquigarrow \textit{ wired arc}) \leq \frac{c_7}{\sqrt{|x-y|n}},$$

where a_n and b_n denote the top-left and top-right corners of the rectangle R_n^β.

Proof. First of all, we note that as in the previous proof, we can take a larger parameter β to widen the rectangle, and this will increase the probability of a vertical crossing. Therefore we may assume, that the parameter β is so big that the points x, y are in the middle of the rectangle, "middle" meaning here that the distance between x and y, $\delta(x,y) = |x-y|$ (δ is the graph metric and $|\cdot|$ is the Euclidean metric of the plane) is strictly smaller than the distance from either of the points x and y to either of the sides of the rectangle; we make this assumption because we will apply Lemma 7.8 and we do not want that the half-ball (the part of the ball that is in the upper half plane) with centre y and radius $|x-y|$ to meet the boundary (also we expect that the half-ball with centre x and radius 1 does not meet the boundary). We shall use the same technique of partition by the exploration path as we did in the previous Lemma 7.8; we use this technique in inequalities (7.25) and (7.26) without presenting the details as we presented them in the proof of Lemma 7.8. Now we note that if x and y are connected to the wired arc in a configuration ω, then the exploration path $\gamma(\omega)$ must pass through the boundary edges of $(R_n^\beta)_\diamond$ that are adjacent to the black diamonds that correspond to x and y. Since x and y are at the middle of the boundary, there are two of these boundary edges for both x and y, the other being an "incoming" and the other an "outgoing" edge; let us call the "incoming" boundary edges e_x and e_y, respectively. Now let us assume that x is to the left of y (the other case is similar); then e_x must be found by the exploration path before e_y, because the exploration path is non-intersecting and because of the topological relations between a_n, x, y and the wired arc and the domain (meaning that the exploration path cannot circle back to e_x once it has gone through e_y when e_y is to the right of e_x).

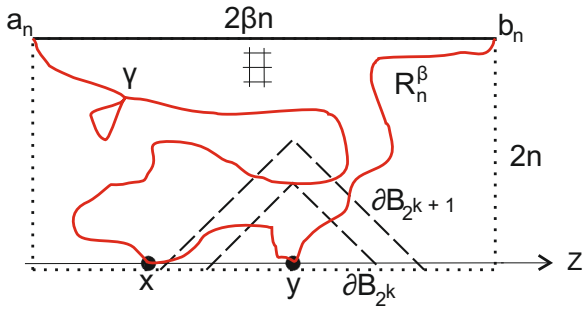


FIGURE 63. The situation in the proof of Lemma 7.9. The exploration path is shown.

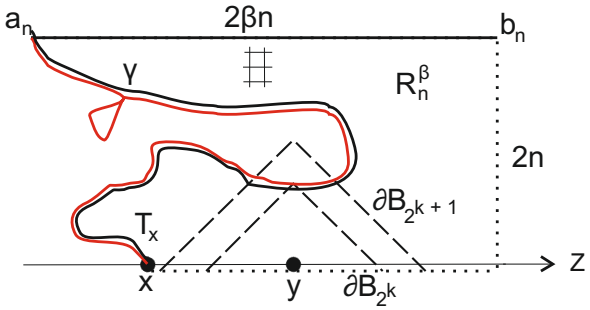


FIGURE 64. In the first step of the proof of Lemma 7.9 we condition with respect to the exploration path at time T_x.

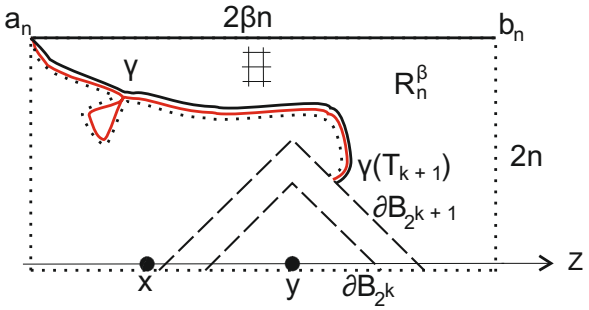


FIGURE 65. In the second step of the proof of Lemma 7.9 we condition with respect to the exploration path at time T_{k+1}.

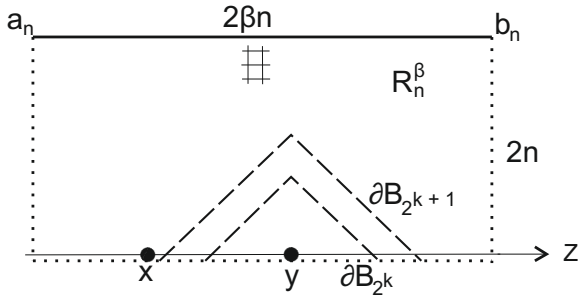


FIGURE 66. In the third step of the proof of Lemma 7.9 we use Lemma 7.8.

Now let us define $T_x(\omega)$ to be the hitting time of the exploration path to e_x with $T_x(\omega) = \infty$ if the exploration path does not go through e_x. Let us also define $T_k(\omega)$ to be the hitting time of the exploration path to the set of medial edges of $(R_n^\beta)_\diamond$ that are adjacent to black squares corresponding to vertexes in the set $B_{2^k}(y) = \{z \in \mathbb{Z}^2 \cap \mathbb{H} \mid \delta(z,y) \le 2^k\}$ for $0 \le k \le k_0 + 1$ for $k_0 = \lfloor \log_2 |x-y| \rfloor$. We set $T_k(\omega) = \infty$ if the exploration path hits e_x before hitting the above defined set of edges; we note that $T_{k_0+1}(\omega) < \infty$ for all ω. With these definitions, we can express the probability of the claim as follows:

(7.25)

$$\begin{aligned}
&P_{(R_n^\beta,a_n,b_n)}(x, y \leftrightsquigarrow \text{ wired arc}) \\
&= P_{(R_n^\beta,a_n,b_n)}(e_x, e_y \in \gamma) \\
&= \sum_{k=0}^{k_o} P_{(R_n^\beta,a_n,b_n)}(\{e_y \in \gamma\} \cap \{T_x < \infty\} \cap \{T_{k+1} < T_k = \infty\}) \\
&= \sum_{k=0}^{k_0} E_{(R_n^\beta,a_n,b_n)}(\chi_{T_{k+1}<T_k=\infty}(\omega)\chi_{T_x<\infty}(\omega) P_{(R_n^\beta,a_n,b_n)}(e_y \in \gamma \,|\, \gamma[0,T_x])(\omega)),
\end{aligned}$$

where we have conditioned on the exploration path up to time T_x. Remember that $e_y \in \gamma(\omega)$ if and only if y is connected to the wired arc in ω.

Now we note that for those configurations ω for which $\chi_{T_{k+1}<T_k=\infty}(\omega)$ $\cdot\chi_{T_x<\infty}(\omega) > 0$ we have that y is at least at graph distance 2^k from the wired arc in $R_n^\beta \setminus \gamma[0,T_x[(\omega)$, since $T_k(\omega) = \infty$. Thus by Domain Markov property and by the first equation in Lemma 7.5 and by Lemma 7.6 we get that

$$\begin{aligned}
&P_{(R_n^\beta,a_n,b_n)}(e_y \in \gamma) \,|\, \gamma[0,T_x])(\omega) \\
&= P_{(R_n^\beta \setminus \gamma[0,T_x[,x,b_n)}(e_y \leftrightsquigarrow \text{ wired arc})(\omega) \\
&\le \frac{c_3}{\sqrt{2^k}}.
\end{aligned}$$

Now we can use the above estimate in equation (7.25), and by noting that $\chi_{T_{k+1}<T_k=\infty}(\omega) = \chi_{T_{k+1}<\infty}(\omega)\chi_{T_k=\infty}(\omega) \le \chi_{T_{k+1}(\omega)<\infty}$, we get that

(7.26)
$$\begin{aligned}
&P_{(R_n^\beta,a_n,b_n)}(x, y \leftrightsquigarrow \text{ wired arc}) \\
&\le \sum_{k=0}^{k_0} E_{(R_n^\beta,a_n,b_n)}(\chi_{T_{k+1}<\infty}(\omega)\chi_{T_x<\infty}(\omega)\frac{c_3}{\sqrt{2^k}}) \\
&\le \sum_{k=0}^{k_0} \frac{c_3}{\sqrt{2^k}} E_{(R_n^\beta,a_n,b_n)}(\chi_{T_{k+1}<\infty}(\omega) P_{(R_n^\beta,a_n,b_n)}(T_x < \infty \,|\, \gamma[0,T_{k+1}])(\omega)),
\end{aligned}$$

in which we have conditioned the path up to time T_{k+1}. Assume ω has $T_{k+1}(\omega) < \infty$ and $k+1 < k_0$; now remember that e_x is in $\gamma(\omega)$ if and only if x is connected to the wired arc, and note that the vertical segment connecting $\gamma(T_{k+1})(\omega)$ to the bottom of the rectangle disconnects the wired arc from x in the domain $R_n^\beta \setminus \gamma[0,T_{k+1}[(\omega)$ because x is to the left of y. The length of

this segment is at most 2^{k+1} (the radius of the ball in graph metric). Now we can use the Domain Markov property and the second part of Lemma 7.5 and Lemma 7.6 to see that

$$\begin{aligned}
&P_{(R_n^\beta,a_n,b_n)}(e_x \in \gamma \,|\, \gamma[0,T_{k+1}])(\omega)\\
&= P_{(R_n^\beta\setminus\gamma[0,T_{k+1}[(\omega),\gamma(T_{k+1})(\omega),b_n)}(x \leftrightsquigarrow \text{ wired arc})\\
&\le 2\frac{\sqrt{c_4 2^{k+1}}}{|x-y|},
\end{aligned}$$

since the vertical segment is at least at distance $\frac{1}{2}|x-y|$ from x for $k+1 < k_0$. Now we use this bound and Lemma 7.8 to estimate

$$\begin{aligned}
&P_{(R_n^\beta,a_n,b_n)}(x, y \leftrightsquigarrow \text{ wired arc})\\
&\le \sum_{k=0}^{k_0-2} \frac{c_3}{\sqrt{2^k}} E_{(R_n^\beta,a_n,b_n)}(\chi_{T_{k+1}<\infty}(\omega) P_{(R_n^\beta,a_n,b_n)}(T_x < \infty \,|\, \gamma[0,T_{k+1}])(\omega))\\
&\quad + \frac{c_3}{\sqrt{2^{k_0-1}}} E_{(R_n^\beta,a_n,b_n)}(\chi_{T_{k_0}<\infty}(\omega) P_{(R_n^\beta,a_n,b_n)}(T_x < \infty \,|\, \gamma[0,T_{k_0}])(\omega))\\
&\quad + \frac{c_3}{\sqrt{2^{k_0}}} E_{(R_n^\beta,a_n,b_n)}(\chi_{T_{k_0+1}<\infty}(\omega) P_{(R_n^\beta,a_n,b_n)}(T_x < \infty \,|\, \gamma[0,T_{k_0+1}])(\omega))\\
&\le \sum_{k=0}^{k_0-2} \frac{c_3}{\sqrt{2^k}} P_{(R_n^\beta,a_n,b_n)}(T_{k+1} < \infty) 2\frac{\sqrt{c_4 2^{k+1}}}{|x-y|} + \frac{c_3}{\sqrt{2^{k_0-1}}} P_{(R_n^\beta,a_n,b_n)}(T_x < \infty)\\
&\quad + \frac{c_3}{\sqrt{2^{k_0-1}}} P_{(R_n^\beta,a_n,b_n)}(T_x < \infty)\\
&\le 2\frac{c_3\sqrt{2c_4}}{|x-y|} \sum_{k=0}^{k_0-2} P_{(R_n^\beta,a_n,b_n)}(T_{k+1} < \infty) + 2\frac{c_3}{\sqrt{2^{k_0-1}}} P_{(R_n^\beta,a_n,b_n)}(T_x < \infty)\\
&\le 2\frac{c_3\sqrt{2c_4}}{|x-y|} \sum_{k=0}^{k_0-2} c_6\sqrt{\frac{2^{k+1}}{n}} + 2\frac{c_3}{\sqrt{2^{k_0-1}}} c_6\sqrt{\frac{1}{n}}\\
&= 4\frac{c_3c_6\sqrt{c_4}}{\sqrt{n}|x-y|}\frac{1-\sqrt{2^{k_0-1}}}{1-\sqrt{2}} + 2\frac{c_3c_6}{\sqrt{n2^{k_0-1}}}\\
&\le 4\frac{c_3c_6\sqrt{c_4}}{\sqrt{n}|x-y|}\frac{\sqrt{2^{k_0-1}}-1}{\frac{4}{3}-1} + 2\frac{c_3c_6}{\sqrt{n2^{k_0-1}}}\\
&\le 12\frac{c_3c_6\sqrt{c_4}}{\sqrt{n}|x-y|}\cdot(|x-y|-1) + 2\frac{c_3c_6}{\sqrt{n\frac{1}{4}|x-y|}}\\
&\le 12\frac{c_3c_6\sqrt{c_4}}{\sqrt{n}}\cdot\frac{1}{\sqrt{|x-y|}} + 2\frac{c_3c_6}{\sqrt{n\frac{1}{4}|x-y|}}\\
&\le (12c_3c_6\sqrt{c_4} + 4c_3c_6)\frac{1}{\sqrt{n|x-y|}} = \frac{c_7}{\sqrt{|x-y|n}},
\end{aligned}$$

where we have used the estimate

$$E_{(R_n^\beta,a_n,b_n)}(\chi_{T_{k+1}<\infty}(\omega)P_{(R_n^\beta,a_n,b_n)}(T_x<\infty\,|\,\gamma[0,T_{k+1}])(\omega))\le P_{(R_n^\beta,a_n,b_n)}(T_x<\infty),$$

which can be seen as follows:

$$\begin{aligned}
&E_{(R_n^\beta,a_n,b_n)}(\chi_{T_{k+1}<\infty}(\omega)P_{(R_n^\beta,a_n,b_n)}(T_x<\infty\,|\,\gamma[0,T_{k+1}])(\omega))\\
&=\sum_{\omega\in\Omega}\chi_{T_{k+1}<\infty}(\omega)P_{(R_n^\beta,a_n,b_n)}(T_x<\infty\,|\,\gamma[0,T_{k+1}])(\omega)P_{(R_n^\beta,a_n,b_n)}(\omega)\\
&=\sum_{\omega\in\{T_{k+1}<\infty\}}P_{(R_n^\beta,a_n,b_n)}(T_x<\infty\,|\,\gamma[0,T_{k+1}])(\omega)P_{(R_n^\beta,a_n,b_n)}(\omega)\\
&=\sum_{\gamma[0,T_{k+1}]}P_{(R_n^\beta,a_n,b_n)}(T_x<\infty\,|\,\gamma[0,T_{k+1}])(\omega)\sum_{\omega\text{ has }\gamma[0,T_{k+1}]}P_{(R_n^\beta,a_n,b_n)}(\omega)\\
&=\sum_{\gamma[0,T_{k+1}]}P_{(R_n^\beta,a_n,b_n)}(T_x<\infty\,|\,\gamma[0,T_{k+1}])(\omega)P_{(R_n^\beta,a_n,b_n)}(\gamma[0,T_{k+1}])\\
&=\sum_{\gamma[0,T_{k+1}]}P_{(R_n^\beta,a_n,b_n)}(\{T_x<\infty\}\cap\{\gamma[0,T_{k+1}]\})\le P_{(R_n^\beta,a_n,b_n)}(T_x<\infty),
\end{aligned}$$

where $P_{(R_n^\beta,a_n,b_n)}(\omega)$ is the weight of the configuration ω. We have expressed the expectation as a sum and manipulated the sum such that instead of summing over appropriate configurations we sum over the different paths that can be considered as $\gamma[0,T_{k+1}]$-exploration paths (on a finite graph there is only a finite amount of these kind of paths since the exploration path is non-self-intersecting, so that there are no problems in defining this sum). We note that the set $\{T_{k+1}<\infty\}$ can be partitioned based on the exploration path $\gamma[0,T_{k+1}]$, and that the set $\{T_x>\infty\}$ on the other hand cannot be fully partitioned based on the exploration path $\gamma[0,T_{k+1}]$, for there are configurations ω in $\{T_x<\infty\}$ that do not have an exploration path $\gamma[0,T_{k+1}](\omega)$ (when $2^{k+1}<|x-y|$) so that we see that the measure of the partitioned part of $\{T_x<\infty\}$ is less than the measure of the whole set (last inequality). □

Now we can prove our main Theorem 7.9:

Proof of main Theorem 7.9. Let $\beta>0$, $n>0$, R_n^β be as above.

First we prove the lower bound for free boundary conditions. First, let us define a random variable $N_n(\omega)$ as the number of pairs $x\in\partial_- R_n^\beta$ and $u\in\partial_+ R_n^\beta$ that are connected by an open path. Let us study the expectation of this random variable:

$$E^0_{R_n^\beta}(N_n)=\sum_{\substack{x\in\partial_- R_n^\beta\\ u\in\partial_+ R_n^\beta}}P^0_{R_n^\beta}(x\leftrightsquigarrow u),$$

where again we see the equality through considering the expectation as a sum over configurations and making rearrangements to the sum. Now we

have a direct lower bound given by Lemma 7.7:

$$\begin{aligned} E^0_{R^\beta_n}(N_n) &\geq \left(2\left\lfloor\frac{\beta n}{2}\right\rfloor + 1\right)^2 \cdot \frac{c_5(\beta)}{n} \\ &\geq \left(2\left(\frac{\beta n}{2} - 1\right) + 1\right)^2 \cdot \frac{c_5(\beta)}{n} \\ &\geq \left(\beta^2 n - 2\beta\right) n \cdot \frac{c_5(\beta)}{n} \\ &= c_5(\beta)\left(\beta^2 n - 2\beta\right), \end{aligned}$$

where we have summed over the $2\left\lfloor\frac{\beta n}{2}\right\rfloor + 1$ ($\left\lfloor\frac{\beta n}{2}\right\rfloor$ points on both sides of the origin and adding the point of origin) points that are far enough from the sides of the rectangle to apply the Lemma 7.7.

Now we note that if x and u (respectively, y and v) are pairwise connected, then they are also connected to the line $\mathbb{Z} \times \{m\}$, where $m = \frac{n}{2}$ if n is even and $m = \frac{n-1}{2}$ if n is odd (note that the case $n = 1$ is redundant), that halves the rectangle R^β_n horizontally such that the "slice" of the rectangle below the line is smaller in the vertical direction than the "slice" above. Now let us denote the event that this line described above is open by L and let us denote the event of a vertical crossing of the rectangle from x to u (y to v) by $\mathrm{CRO}(x, u)$ ($\mathrm{CRO}(y, v)$). We note that L and $\mathrm{CRO}(x, y)$ ($\mathrm{CRO}(y, v)$) are increasing events, so by FKG inequality

$$P^0_{R^\beta_n}(L \cap \mathrm{CRO}(x, u)) \geq P^0_{R^\beta_n}(L) P^0_{R^\beta_n}(\mathrm{CRO}(x, u)).$$

Therefore we have that

$$P^0_{R^\beta_n}(\mathrm{CRO}(x, u)) \leq \frac{P^0_{R^\beta_n}(L \cap \mathrm{CRO}(x, u))}{P^0_{R^\beta_n}(L)} = P^0_{R^\beta_n}(\mathrm{CRO}(x, u) \,|\, L).$$

Now let us denote the event of x being connected to the "centre" line from below by B_x and the event that u is connected to the "centre" line from above by A_u (B_y and A_v, respectively). Then we have

$$\begin{aligned}
P^0_{R^\beta_n}(\mathrm{CRO}(x,u)) &\leq P^0_{R^\beta_n}(\mathrm{CRO}(x,u)\,|\,L) \\
&\leq P^0_{R^\beta_n}(B_x \cap A_u \,|\, L) \\
&= \frac{P^0_{R^\beta_n}(B_x \cap A_u \cap L)}{P^0_{R^\beta_n}(L)} \\
&= \frac{P^0_{R^\beta_n}(B_x \cap L) \cdot P^0_{R^\beta_n}(A_u \cap L)}{P^0_{R^\beta_n}(L)} \\
&\leq \frac{1}{P^0_{R^\beta_n}(L)} \frac{P^0_{R^\beta_n}(B_x \cap L) \cdot P^0_{R^\beta_n}(A_u \cap L)}{P^0_{R^\beta_n}(L)} \\
&= P^0_{R^\beta_n}(B_x \,|\, L) P^0_{R^\beta_n}(A_u \,|\, L)
\end{aligned}$$

because we note that B_x and A_u are independent events as one can see by concerning the Domain Markov property such that the domains are cut along the "centre" line and the "centre" line is conditioned open, so that no more connections than what is conditioned (since we condition "all connections" by conditioning the boundary line open) can happen in the part of the domain that is cut out (similarly for y, v, $\mathrm{CRO}(y,v)$, B_y and A_v). Now we note that since the slice of the domain under the "centre" line is smaller or equal in vertical size to the upper part of the domain, $P^0_{R^\beta_n}(A_u \,|\, L) \leq P^0_{R^\beta_n}(B_{\overline{u}} \,|\, L)$, where we denote by $\overline{u}$ the projection of u to the real line, for the upper and lower parts of the domain are in other respects but the size similar domains, so random cluster percolation in both of them is a similar process (we can obviously relate the event A_u to an event $A'_u \subset B_{\overline{u}}$). Now for last we note that considering B_x, $B_{\overline{u}}$ and conditioning on L can be regarded (through Domain Markov property) as studying the events B_x, $B_{\overline{u}}$ in the domain that is the lower part of the rectangle with wired boundary conditions replacing the "centre" line. Hence the probability that x, y, u and v are connected to the "centre" line in R^β_n with free boundary conditions is smaller that the probability of x, y, u and v being connected to the boundary that replaces the "centre" line in a domain that is the rectangle $[-\beta n, \beta n] \times [0, m] = \frac{1}{2} V R^\beta_n$, that is, a rectangle that is no more than half the height of the original rectangle, such that this new domain has wired boundary conditions on the top side. Note that the right hand side of the above equality (concerning the "half-height rectangle"-domain) is simply the definition of the denotation $\frac{1}{2} V R^\beta_n$. Now letting $a_{\frac{1}{2}V}$ and $b_{\frac{1}{2}V}$ be the top

left and top right corners of the rectangle $\frac{1}{2}VR_n^\beta$, we get

$$P^0_{R_n^\beta}(\{x \leftrightsquigarrow u\} \cap \{y \leftrightsquigarrow v\}) \leq P_{(\frac{1}{2}VR_n^\beta, a_{\frac{1}{2}V}, b_{\frac{1}{2}V})}(x, y \leftrightsquigarrow \text{ wired arc}) \cdot P_{(\frac{1}{2}VR_n^\beta, a_{\frac{1}{2}V}, b_{\frac{1}{2}V})}(\overline{u}, \overline{v} \leftrightsquigarrow \text{ wired arc}).$$

Now we use Lemma 7.9 to get a bound for the above probability when $x \neq y$, $u \neq v$:

$$P^0_{R_n^\beta}(\{x \leftrightsquigarrow u\} \cap \{y \leftrightsquigarrow v\}) \leq \frac{c_7}{\sqrt{|x-y| \cdot \frac{1}{2}n}} \cdot \frac{c_7}{\sqrt{|\overline{u}-\overline{v}| \cdot \frac{1}{2}n}}.$$

In the case when $x = y$, $u \neq v$, we get similarly (using a trivial upper bound of 1 in the $\{x, y \leftrightsquigarrow \text{ wired arc}\}$ probability) an upper bound for the above mentioned probability:

$$P^0_{R_n^\beta}(\{x \leftrightsquigarrow u\} \cap \{y \leftrightsquigarrow v\}) \leq \frac{c_7}{\sqrt{|\overline{u}-\overline{v}| \cdot \frac{1}{2}n}};$$

in the case $x \neq y$, $u = v$ we get similarly an upper bound

$$P^0_{R_n^\beta}(\{x \leftrightsquigarrow u\} \cap \{y \leftrightsquigarrow v\}) \leq \frac{c_7}{\sqrt{|x-y| \cdot \frac{1}{2}n}}$$

and finally in the case $x = y$, $u = v$ we can use the upper bound

$$P^0_{R_n^\beta}(\{x \leftrightsquigarrow u\} \cap \{y \leftrightsquigarrow v\}) \leq 1.$$

Now we sum over all x,$y \in \partial_- R_n^\beta$ and u, $v \in \partial_+ R_n^\beta$ to obtain

$$E^0_{R_n^\beta}(N_n^2) = \sum_{\substack{x,y \in \partial_- R_n^\beta \\ u,v \in \partial_+ R_n^\beta}} P^0_{R_n^\beta}(\{x \leftrightsquigarrow u\} \cap \{y \leftrightsquigarrow v\}),$$

where again, we imagine the expectation as a finite sum and rearrange the configurations ω that appear as terms $N_n^2(\omega)P^0_{R_n^\beta}(\omega)$ in the sum into new order (we imagine that each configuration is weighted by the number N_n^2 and then we make sure that N_n^2 copies of the same configuration appear on the right hand side in the probabilities; the probability of an event is simply the finite sum of the probabilities of the configurations in that event). Note that if a configuration ω has N distinct ordered pairs (x, u), $x \in \partial_- R_n^\beta$ and $u \in \partial_+ R_n^\beta$ that are connected, then the configuration receives the weight N^2 in the "expectation side", and in the "probability side", the configuration appears in N^2 probabilities $P^0_{R_n^\beta}(\{x \leftrightsquigarrow u\} \cap \{y \leftrightsquigarrow v\})$, namely the "$x$- and u-coordinates" in the probability can be chosen in N ways (any of the connected, distinct pairs will do), and the "y- and v-coordinates" can also be chosen in N ways. Now we shall utilize the bounds above (note:

$|\overline{u} - \overline{v}| = |u - v|$)

$$
\begin{aligned}
&E^0_{R_n^\beta}(N_n^2) \\
&\leq \sum_{\substack{x,y\in\partial_- R_n^\beta \\ u,v\in\partial_+ R_n^\beta \\ x\neq y \\ u\neq v}} \frac{2c_7^2}{n\sqrt{|x-y||u-v|}} + \sum_{\substack{x=y\in\partial_- R_n^\beta \\ u,v\in\partial_+ R_n^\beta \\ u\neq v}} \frac{c_7}{\sqrt{|u-v|\cdot\frac{1}{2}n}} + \sum_{\substack{x,y\in\partial_- R_n^\beta \\ u=v\in\partial_+ R_n^\beta \\ x\neq y}} \frac{c_7}{\sqrt{|x-y|\cdot\frac{1}{2}n}} + 1 \\
&\leq \frac{2c_7^2}{n} \sum_{\substack{x,y\in\partial_- R_n^\beta \\ x\neq y}} \frac{1}{\sqrt{|x-y|}} \sum_{\substack{u,v\in\partial_+ R_n^\beta \\ u\neq v}} \frac{1}{\sqrt{|u-v|}} + 2\cdot 2\beta n \frac{2c_7^2}{n} \sum_{\substack{x,y\in\partial_- R_n^\beta \\ x\neq y}} \frac{1}{\sqrt{|x-y|}} + n^2 \\
&= \frac{2c_7^2}{n} \left(\sum_{\substack{x,y\in\partial_- R_n^\beta \\ x\neq y}} \frac{1}{\sqrt{|x-y|}} \right)^2 + 8\beta c_7^2 \sum_{\substack{x,y\in\partial_- R_n^\beta \\ x\neq y}} \frac{1}{\sqrt{|x-y|}} + n^2,
\end{aligned}
$$

where we note that $\partial_- R_n^\beta$ and $\partial_+ R_n^\beta$ have equal amount of members and that the sums over those sets are actually equal. Now let us study the sum $\sum_{x,y\in\partial_- R_n^\beta} \frac{1}{\sqrt{|x-y|}}$:

$$
\sum_{\substack{x,y\in\partial_- R_n^\beta \\ x\neq y}} \frac{1}{\sqrt{|x-y|}} = \sum_{x\in\partial_- R_n^\beta} \sum_{\substack{y\in\partial_- R_n^\beta \\ y\neq x}} \frac{1}{\sqrt{|x-y|}} \leq 2\beta n \sum_{j=1} \frac{1}{\sqrt{j}}
$$

because for every x we can approximate the y-sum from the above with the j-sum, and there are $2\beta n$ choices of x. Now we finally can approximate

$$
\sum_{j=1} \frac{1}{\sqrt{j}} \leq \int_0^{2\beta n} \frac{1}{\sqrt{x}} dx = \sqrt{2\beta n} - \sqrt{0} = \sqrt{2\beta}\sqrt{n}
$$

and hence we get:

$$
\begin{aligned}
E^0_{R_n^\beta}(N_n^2) &\leq \frac{2c_7^2}{n}(2\beta n\sqrt{2\beta}\sqrt{n})^2 + 8\beta c_7^2 \cdot 2\beta n\sqrt{2\beta}\sqrt{n} + n^2 \\
&\leq 16c_7^2\beta^3 n^2 + 16\sqrt{2}\beta^2\sqrt{\beta}c_7^2 n^2 + n^2 \leq c_8(\beta)n^2
\end{aligned}
$$

for some $c_8(\beta) > 0$.

Now we can use the Cauchy-Schwarz inequality

$$
\begin{aligned}
P^0_{R_n^\beta}(V(R_n^\beta)) &= P^0_{R_n^\beta}(N_n > 0) \\
&= E^0_{R_n^\beta}(\chi^2_{N_n>0}) \\
&\geq \frac{E^0_{R_n^\beta}(N_n)^2}{E^0_{R_n^\beta}(N_n^2)} \\
&\geq \frac{\left(c_5(\beta)\left(\beta^2 n - 2\beta\right)\right)^2}{n^2},
\end{aligned}
$$

where the Cauchy-Schwarz inequality is applied as

$$E^0_{R_n^\beta}(N_n) = E^0_{R_n^\beta}(N_n \chi_{N_n>0}) \leq \left(E^0_{R_n^\beta}(N_n^2)\right)^{\frac{1}{2}} \left(E^0_{R_n^\beta}(\chi^2_{N_n>0})\right)^{\frac{1}{2}}.$$

Thus we have the claim of the lower bound for free boundary conditions; since the boundary conditions are monotonous, we can deduce that the lower bound holds for any boundary condition ξ.

Then we shall show the upper bound. Because of the monotonicity of the boundary conditions, we see that it is enough to show the upper bound for the wired boundary condition, denoted with a superscript 1. Now we shall utilize the idea of duality: the probability that in the dual model there exist a dual horizontal crossing from the left side to the right side is bounded below by the above, since the dual of the rectangle with wired boundary conditions is a (dual lattice) rectangle with free boundary conditions, and we have chosen p to be self-dual, so that the primal and dual random cluster models are equivalent; hence we can use the above argumentation in the dual model. Note that if the relation of the width of the rectangle to the height of the rectangle is β when we study the vertical crossing, then when we change our attention to the horizontal crossing, we need only to change to above ratio to $\frac{1}{\beta}$ since the lattice is invariant in $\frac{\pi}{2}$-rotations, and we can continue studying the vertical crossing in this new $\frac{1}{\beta}$-ratio rectangle. Or to put it more simply, the horizontal crossing in the rectangle with aspect ratio β is as probable as a vertical crossing in a rectangle with aspect ratio $\frac{1}{\beta}$.

Hence the probability of the dual horizontal crossing is bounded below by $c(\frac{1}{\beta})$, and we deduce

$$\begin{aligned} P^\xi_{R_n^\beta}(V(R_n^\beta)) &\leq P^1_{R_n^\beta}(V(R_n^\beta)) \\ &\leq 1 - P^1_{R_n^\beta}(H_d(R_n^\beta)) \\ &= 1 - P^0_{(R_n^\beta)_d}(H((R_n^\beta)_d)) \\ &\leq 1 - c(\frac{1}{\beta}), \end{aligned}$$

where we denote $H_d(R_n^\beta) = \{$configurations ω of R_n^β such that in ω there is no possibility to make a horizontal dual crossing of $(R_n^\beta)_d\}$. Thus we have proven our main theorem. □

8. Application of box-crossing property: an example of an arm exponent

Next we shall show an application of the box-crossing property to a so-called *arm exponent*.

8.1. **Half-plane one-arm exponent.**

Theorem 8.1. *Let us consider the rectangle $R_n = [-n, n] \times [0, n]$. There exists positive constants $0 < d_1 \leq d_2$ such that for any boundary conditions ξ such that the bottom boundary $\partial_- R$ is free it holds that*

$$d_1 n^{-\frac{1}{2}} \leq P^{\xi}_{R_n}(0 \leftrightsquigarrow \partial^+ R_n) \leq d_2 n^{-\frac{1}{2}}$$

uniformly in n, where the constant $\frac{1}{2}$ is the half-plane one-arm exponent *and the event $0 \leftrightsquigarrow \partial^+ R_n$ is the* half-plane one-arm event*, and $\partial_+ R$ denotes the top boundary of the rectangle.*

Proof. By monotonicity of boundary conditions it is enough for upper bound to study the case where the boundary conditions are such that the sides of the rectangle other than the bottom side have wired boundary conditions. Then we just trivially note that the probability of existence of an open connection between 0 and $\partial^+ R$ is less than the probability of existence of an open connection between 0 and the whole of the wired arc (when wired arc is defined as above to cover the whole boundary of the rectangle except the bottom boundary). Then we can use Lemma 7.6 to compare the probability of the origin being connected to the wired arc with the square root of the harmonic measure of the wired arc $\mathrm{HM}(B_0)$. Now the first part of the Lemma 7.5 gives us the upper bound.

For the lower bound, by the box-crossing property and the FKG-inequality, we can as before in the percolation-subsections (using three rectangles inside the half-annulus introduced below) deduce that at probability greater than a positive constant a which is independent of ξ and n, there is an open "half-circle", let us denote both this half-circle and the event that such an open half-circle exists by γ, in the half-annulus $R_n \setminus R_{\lfloor \frac{n}{2} \rfloor}$ disconnecting the origin from the set $\partial R_n \setminus \partial_- R_n$ (where by ∂R_n we mean the set of vertexes at the geometrical boundary of the rectangle); see figure 67. Now we claim that the probability that 0 is connected to this "half-circle" by an open path, conditioned by the existence of the "half-circle" is larger than the probability that the origin is connected to the wired boundary in the rectangle R_n with wired boundary conditions on $\partial R_n \setminus \partial_- R_n$, let us denote these boundary conditions with ζ, without conditioning. This can be seen as follows:

$$\begin{aligned} P^{\xi}_{R_n}(0 \leftrightsquigarrow \gamma \,|\, \gamma) &= P^{\zeta}_{R_n}(0 \leftrightsquigarrow \gamma \,|\, \gamma) \\ &\geq P^{\zeta}_{R_n}(0 \leftrightsquigarrow \partial R_n \setminus \partial_- R_n \,|\, \gamma) \\ &\geq P^{\zeta}_{R_n}(0 \leftrightsquigarrow \partial R_n \setminus \partial_- R_n) \end{aligned}$$

where the first equality comes simply from the Domain Markov property, or to be more precise, from the domain slicing property introduced above,

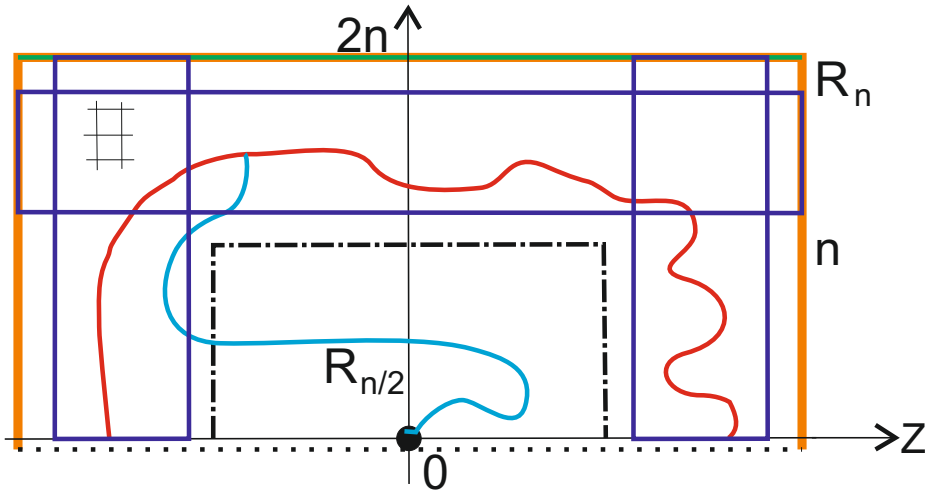


FIGURE 67. The proof of Theorem 8.1. The "half-circle" inside the "half-annulus" $R_n \setminus R_{\lfloor \frac{n}{2} \rfloor}$ in red and an open connection from the origin to it in light blue; the blue rectangles are the rectangles used in connection with the box-crossing property to show that the half-circle exits with positive probability. We denote the boundary conditions ξ by the orange line and $\partial^+ R$ by the green line.

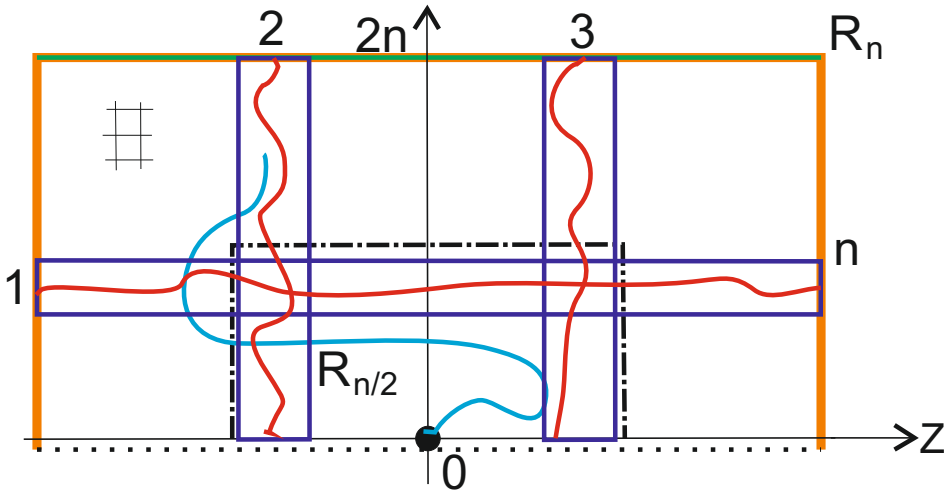


FIGURE 68. The proof of Theorem 8.1. The situation with open crossing from 0 to $R_n \setminus R_{\lfloor \frac{n}{2} \rfloor}$ shown in light blue, and the open crossings of the blue rectangles 1 2 and 3 are in red.

since the half-circle γ slices the domain R_n into two independent parts, and the events we consider happen only in the lower part of the domain. The next inequality simply follows from the fact that we demand more (larger open connection) on the right hand side than on the left hand side. We also

use FKG inequality:

$$P^{\zeta}_{R_n}(\{0 \leftrightsquigarrow \partial R_n \setminus \partial_- R_n\} \cap \{\gamma\}) \geq P^{\zeta}_{R_n}(0 \leftrightsquigarrow \partial R_n \setminus \partial_- R_n) P^{\zeta}_{R_n}(\gamma)$$
$$\Rightarrow P^{\zeta}_{R_n}(0 \leftrightsquigarrow \partial R_n \setminus \partial_- R_n \,|\, \gamma) \geq P^{\zeta}_{R_n}(0 \leftrightsquigarrow \partial R_n \setminus \partial_- R_n).$$

Now we can again use Lemma 7.6 to relate the probability

$$P^{\zeta}_{R_n}(0 \leftrightsquigarrow \partial R_n \setminus \partial_- R_n) = P^{\zeta}_{R_n}(0 \leftrightsquigarrow \text{ wired arc})$$

to the square root of the harmonic measure $\mathrm{HM}(W_0)$, and then the Lemma 7.4 gives us the lower bound, since obviously the segment $L_n(-n)$ separates the origin from the part of the dual free arc above the horizontal axis, so that

$$P^{\zeta}_{R_n}(0 \leftrightsquigarrow \text{ wired arc}) \geq \sqrt{\mathrm{HM}(W_0)} \geq \sqrt{\frac{c_2}{n}}.$$

Now we put all the pieces together to find:

$$\begin{aligned}
\sqrt{\frac{c_2}{n}} &\leq P^{\zeta}_{R_n}(0 \leftrightsquigarrow \text{ wired arc}) \\
&\leq P^{\xi}_{R_n}(0 \leftrightsquigarrow \gamma \,|\, \gamma) \\
&\leq \frac{P^{\xi}_{R_n}(\gamma)}{a} P^{\xi}_{R_n}(0 \leftrightsquigarrow \gamma \,|\, \gamma) \\
&= \frac{1}{a} P^{\xi}_{R_n}(\{0 \leftrightsquigarrow \gamma\} \cap \{\gamma\}) \\
&\leq \frac{1}{a} P^{\xi}_{R_n}(0 \leftrightsquigarrow R_n \setminus R_{\lfloor \frac{n}{2} \rfloor}).
\end{aligned}$$

Now let us relate this final probability $P^{\xi}_{R_n}(0 \leftrightsquigarrow R_n \setminus R_{\lfloor \frac{n}{2} \rfloor})$ to the originally desired probability $P^{\xi}_{R_n}(0 \leftrightsquigarrow \partial^+ R_n)$; to do this, let as introduce three rectangles 1,2 and 3 as in figure 68, and constants (independent of n and in ξ) $a_i > 0$, $i = 1, 2, 3$, which are the lower bounds from the box-crossing property for the side-to-side crossings of the rectangles $1, 2, 3$ depicted in figure 68, that is, for the horizontal crossing in rectangle 1 and for the vertical crossing in rectangles 2 and 3. Let us denote the events of the above described crossing taking place in rectangle $i = 1, 2, 3$ by $C(i)$, and let us note that if all three crossings and the event $\{0 \leftrightsquigarrow R_n \setminus R_{\lfloor \frac{n}{2} \rfloor}\}$ take place, then we have $\{0 \leftrightsquigarrow \partial^+ R_n\}$. Then, by using the FKG inequality ($C(i)$, $i = 1, 2, 3$ and $\{0 \leftrightsquigarrow R_n \setminus R_{\lfloor \frac{n}{2} \rfloor}\}$ are increasing events) we get that

$$\begin{aligned}
&P^{\xi}_{R_n}(0 \leftrightsquigarrow R_n \setminus R_{\lfloor \frac{n}{2} \rfloor}) \\
&\leq \frac{P^{\xi}_{R_n}(C(1))}{a_1} \cdot \frac{P^{\xi}_{R_n}(C(2))}{a_2} \cdot \frac{P^{\xi}_{R_n}(C(3))}{a_3} \cdot P^{\xi}_{R_n}(0 \leftrightsquigarrow R_n \setminus R_{\lfloor \frac{n}{2} \rfloor}) \\
&\leq \frac{1}{a_1 \cdot a_2 \cdot a_3} P^{\xi}_{R_n}(0 \leftrightsquigarrow \partial^+ R_n),
\end{aligned}$$

Thus we see that we have the desired lower bound. □

Appendix A. Maximum principle for discrete harmonic functions

We prove a discrete version of the familiar theorem of complex analysis or real harmonic function theory, namely the *maximum principle*; a subharmonic (or in particular, harmonic) function reaches its maximum at the boundary, and if the function attains its global maximum on some interior point, then the function is constant in the interior (see below for definition of "interior" and "boundary"). The proof we present applies in any such connected finite discrete domain of the square lattice that the interior sites are connected among themselves (which is the generality which we need in this thesis), but more general statements of similar sorts can also be proven.

Let $V \subset \mathbb{Z}^2$ be such that the square lattice subgraph whose vertex set is V and that has all the edges of $\mathbb{L}^2$ connecting vertexes of V present, let us denote this square lattice domain also by V, is a connected finite discrete domain of the square lattice such that the interior sites are connected among themselves, id est, all interior sites can be connected to all other interior sites by graph-paths that only step from one interior site to an adjacent interior site; here by *interior site* we mean a site that has 4 adjacent sites in V. Those sites that are not interior sites are *boundary sites*. Let $f : V \to \mathbb{R}$ be a function. Let $r : V \to]0, \infty[$ be called *weight function*. The function f is *subharmonic* for a *r-weighted discrete Laplacian* ∇_r^2 if for a lattice site of $S \in V$ that has four sites adjacent in V, let these sites be $S_N, S_W, S_S, S_E \in V$, f satisfies

$$\nabla_r^2 f(S) = \frac{r(S_N)f(S_N) + r(S_W)f(S_W) + r(S_S)f(S_S) + r(S_E)f(S_E)}{r(S_N) + r(S_W) + r(S_S) + r(S_E)} - f(S) \geq 0.$$

The function f is *superharmonic* if $-f$ is subharmonic, and it is *harmonic* if it is both subharmonic and superharmonic (here we mean always with respect to ∇_r^2-Laplacian).

We shall collect all the maximum/minimum principle results we have to the following Lemma A.1 to facilitate referencing into these results; we shall use the denotation introduced above. Note that usually our weights $r(v)$, $v \in V$ are just constant $\frac{1}{4}$, but we need this more general formulation in Theorem 7.12.

Lemma A.1. *Let V be as described above. Let $f : V \to \mathbb{R}$. Let $r : V \to]0, \infty[$ and the r-weighted Laplacian ∇_r^2 be as above. In this Lemma we always refer to this r-weighted Laplacian.*

*(*Maximum principle*) Let f be subharmonic (or in special case harmonic). Then f reaches its maximum at the boundary of V, and if f reaches its global maximum at an interior site $S \in V$, then f is constant in the interior. Furthermore one can relax the assumption of interior sites being connected among themselves, and the maximum will still be attained at the boundary.*

*(*Minimum principle*) Let f is superharmonic (or in special case harmonic). Then f reaches its minimum at the boundary of V, and if f reaches*

its global minimum at an interior site $S \in V$, then f is constant in the interior. Furthermore one can relax the assumption of interior sites being connected among themselves, and the minimum will still be attained at the boundary.

Proof. Let f be subharmonic. Let f reach its global maximum at an interior site $S \in V$. We prove that f is constant in the interior. Because S is in the interior, f is subharmonic at S; let us denote the neighbouring sites of S by S_N, S_W, S_S, S_E:

$$\nabla_r^2 f(x) = \frac{r(S_N)f(S_N) + r(S_W)f(S_W) + r(S_S)f(S_S) + r(S_E)f(S_E)}{r(S_N) + r(S_W) + r(S_S) + r(S_E)} - f(S) \geq 0$$

or equivalently

$$\frac{r(S_N)f(S_N) + r(S_W)f(S_W) + r(S_S)f(S_S) + r(S_E)f(S_E)}{r(S_N) + r(S_W) + r(S_S) + r(S_E)} \geq f(S). \tag{A.1}$$

From this inequality we see that if A is a neighbour (one of those neighbours if there are many of them) of S that has the greatest f-value, then $f(A) \geq f(S)$, for if all the neighbours of S have f-value lower than S, then the left hand side of (A.1) is too small; furthermore, if $f(A) = f(S)$, then $f(S_i) = f(S)$ for all $i = N, W, S, E$. But now since f attained a maximum at S, it must be the case that $f(A) = f(S)$, also the function f has the value $f(S)$ on all the neighbours of S. Furthermore, we can iterate this argument for those neighbours of S that are not in the boundary, and see that the function f must have the value $f(S)$ on all the neighbours of neighbours of S (we just take one of the neighbours of S in place of S in the above argument concerning neighbours of S), and on all the neighbours of neighbours of neighbours of S, et cetera. Continuing this, we see that the function f must actually be a constant in the interior of our finite domain (here we use the fact that because the domain is "interior connected", one can walk from one interior site to all the other interior sites by stepping from one interior site to another).

Let us continue study the above argument a little more. Note that since the domain is finite, there must be boundary sites, and because the domain is connected, there must be some interior sites that are connected to boundary sites. When we apply the above argument and start to "travel" from the global maximum interior site S, travelling from one neighbour to the next, then when we "arrive" to a site that is not a boundary site but whose neighbouring site is a boundary site, then we can once more apply our argument to this "next-to-boundary" site and see that the function must attain the value $f(S)$ also at the boundary. Therefore, since $f(S)$ was the global maximum, we have shown that if f reaches its global maximum at the interior, then it also attains this global maximum at the boundary. Note that for this argument to work, id est, for the interior maximum to "migrate" to the boundary, one can relax the assumption that V is such that interior sites are connected among themselves; it is enough that all interior sites of V can be connected to some boundary site, which is always true in a finite connected

domain. This explains the remark made above about relaxing the domain regularity assumption, as can be seen below.

Now the other claim about a subharmonic f reaching its maximum in the boundary is clear; since our domain is finite, f much reach a global maximum in it; if this happens at a site that is not on the boundary, also at a site that is an interior site, then as shown above f reaches its maximum also at the boundary (and the domain regularity assumptions can be relaxed, as noted above); if attaining a maximum does not happen at a site that is not in boundary, then it happens in the boundary (note that in boundary sites f is not subharmonic, or to be more precise, equation (A.1) does not hold and we cannot argue as above that f is a constant).

The claims concerning superharmonic function f follow by applying the maximum principle to $-f$, which is subharmonic since f is superharmonic; then the claim follows just by noticing that the maximum of $-f$ corresponds to the minimum of f. □

Appendix B. Simple random walk on the square lattice in a rectangular domain

Let us study a simple random walk (SRW) in the square lattice, and let us consider in particular the case where the walk happens inside a rectangle R whose sides are parallel with the square lattice axes, and whose height (the direction chosen here is arbitrary, but once the direction of "up" is chosen, one must respect that choice) is N and whose width is aN with $0 < a < \infty$. Let us divide the boundary ∂R of the rectangle, consisting of the boundary vertexes (those vertexes having less than 4 neighbours in the rectangle) into 4 parts: the right $\mathfrak{R}$, top $\mathfrak{T}$, left $\mathfrak{L}$ and bottom $\mathfrak{B}$ boundary, $\mathfrak{R}, \mathfrak{T}, \mathfrak{L}, \mathfrak{B} \subset \partial R$ in the natural way (the corner vertexes are included in $\mathfrak{R}$ and $\mathfrak{L}$ only). Let us also fix our coordinates so that the lower left corner of the rectangle is the origin, and x-coordinate is directed to right and y-coordinate is directed up. Let us finally remind the reader that a simple random walk, let us call it S, is simply a stochastic (that is, random) process in which a walker, starting from some point of the lattice, takes one step to some neighbouring lattice site with equal probabilities once every "time-interval", that is, the state of the process is represented by a sequence $(S_n)_{n\in\mathbb{N}}$ of lattice points following the above rule of equal probability of steps to neighbouring sites; we interpret the index $n \in \mathbb{N}$ as discrete time parameter, so that increasing the index by one represents "an instant of time occurring" during which the walker takes one step. Let us also define a denotation S^v for the simple random walker starting at lattice site $v \in V(R)$.

First, let us prove a small lemma as follows:

Lemma B.1. *Let $a > 2$. There exists constants d_1, $d_2 > 0$ such that for the random walker $S^{(x_0,1)}$ starting in the width-wise middle of the rectangle, that is, at $x_0 = \left\lfloor \frac{aN}{2} \right\rfloor$, one lattice step above the bottom of the rectangle, the probability of hitting the boundary of the rectangle first time at the $\mathfrak{T}$ satisfies:*

$$\frac{d_1}{N} \leq P(\{S^{(x_0,1)} \textit{ hits } \mathfrak{T} \textit{ before } \mathfrak{R} \cup \mathfrak{L} \cup \mathfrak{B}\}) \leq \frac{d_2}{N}.$$

Proof. Let us introduce a new function

$$A(x, y) = P(\{S^{(x,y)} \text{ hits } \mathfrak{T} \text{ before } \mathfrak{R} \cup \mathfrak{L} \cup \mathfrak{B}\}),$$

where (x, y) is a given lattice point in the rectangle. First we note that this function is discrete harmonic, id est,

$$\begin{aligned} &\nabla^2 A(x, y) \\ &= \frac{1}{4}\left(A(x, y+1) + A(x-1, y) + A(x, y-1) + A(x+1, y)\right) - A(x, y) \\ &= 0 \end{aligned} \tag{B.1}$$

on any interior lattice point (x, y) of R (interior points are those that have 4 neighbours in the rectangle). Furthermore we see that A satisfies $A(x, y) = 0$ on lattice points $(x, y) \in \mathfrak{R} \cup \mathfrak{L} \cup \mathfrak{B}$, and $A(x, y) = 1$ for all $(x, y) \in \mathfrak{T}$.

Now let us note that the function $f(x,y) = \frac{y}{N}$ is obviously discrete harmonic (direct calculation) and it has values 0 in $\mathfrak{B}$ and 1 in $\mathfrak{T}$, and $0 \leq f(x,y) \leq 1$ in $\mathfrak{R} \cup \mathfrak{L}$, so that $f \geq A$ on the boundary of R, and since f and A are both discrete harmonic, by the maximum principle of (discrete) harmonic functions which was shown in the appendix Lemma A.1, $f \geq A$ on the whole R. Especially $f(x,1) = \frac{1}{N} \geq A(x,1)$, so actually $d_2 = 1$ (is one possible choice).

Let us also define

$$B(x,y) = P(\{S^{(x,y)} \text{ hits } \mathfrak{R} \cup \mathfrak{T} \cup \mathfrak{L} \text{ before } \mathfrak{B}\}).$$

Again, this function is discrete harmonic inside R, and it satisfies $B(x,y) = 1$ in $\mathfrak{R} \cup \mathfrak{T} \cup \mathfrak{L}$, and $B(x,y) = 0$ in $\mathfrak{B}$. Therefore $B \geq f$ on the boundary of R and therefore in the whole R. Especially $B(x,1) \geq f(x,1) = \frac{1}{N}$.

Let us then introduce the functions

$$C(x,y) = P(\{S^{(x,y)} \text{ hits } \mathfrak{R} \cup \mathfrak{L} \text{ before } \mathfrak{T} \cup \mathfrak{B}\})$$

and $g(x,y) = \frac{1}{\left(\frac{aN}{2}\right)^2}\left(\left(x - \frac{aN}{2}\right)^2 - y^2 + Ny\right)$. These functions are also discrete harmonic, as can be seen by direct calculation (g is actually the sum of the real part of the analytic complex function $z = x + iy \mapsto \frac{1}{\left(\frac{aN}{2}\right)^2}\left(z - \frac{aN}{2}\right)^2$, the real part being (discrete) harmonic, and the obviously discrete harmonic function $\frac{1}{\left(\frac{aN}{2}\right)^2}Ny$). In the boundaries C satisfies: $C(x,y) = 0$ in $\mathfrak{T} \cup \mathfrak{B}$, and $C(x,y) = 1$ in $\mathfrak{R} \cup \mathfrak{L}$, and g satisfies $g(x,y) = \frac{1}{\left(\frac{aN}{2}\right)^2}x^2$ in $\mathfrak{T} \cup \mathfrak{B}$, and $g(x,y) \geq 1$ in $\mathfrak{R} \cup \mathfrak{L}$(as in $\mathfrak{R} \cup \mathfrak{L}$ we have $x - \frac{aN}{2} = \pm\frac{aN}{2}$ and $-y^2 + Ny \geq 0$). Therefore, by maximum principle of (discrete) harmonic functions, since $g \geq C$ on the boundaries, $g \geq C$ on the whole domain R. Especially $C\left(\left\lfloor \frac{aN}{2} \right\rfloor, 1\right) \leq g\left(\left\lfloor \frac{aN}{2} \right\rfloor, 1\right) \leq \frac{1+N-1}{\left(\frac{aN}{2}\right)^2} = \frac{4}{a^2N}$.

Now we deduce:

$$\begin{aligned}
&P(\{S^{(x_0,1)} \text{ hits } \mathfrak{T} \text{ before } \mathfrak{R} \cup \mathfrak{L} \cup \mathfrak{B}\}) \\
=&P(\{S^{(x_0,1)} \text{ hits } \mathfrak{R} \cup \mathfrak{T} \cup \mathfrak{L} \text{ before } \mathfrak{B}\}) - P(\{S^{(x_0,1)} \text{ hits } \mathfrak{R} \cup \mathfrak{L} \text{ before } \mathfrak{T} \cup \mathfrak{B}\}) \\
\geq&\frac{1}{N} - \frac{4}{a^2N} = \frac{a^2-4}{a^2N},
\end{aligned}$$

so that $d_1 = \frac{a^2-4}{a^2}$ (is one possible choice). □

Next we shall prove the following result: starting in the middle of a rectangle, the probability that a random walker exits the rectangle through a given boundary side (named below the top boundary, but because of symmetry any boundary side will do) is bounded above zero by a positive constant that does not depend on the size of the rectangle (although it can depend on the aspect ratio). We will give a more formal formulation below.

Lemma B.2. *For all $N \in \mathbb{N}$, for all $0 < b <\leq 1$, there exists a $d_3 = d_3(a,b) > 0$ (note: uniform in N) such that the probability that a random walker $S^{(x,\lfloor bN \rfloor)}$ that starts at height $\lfloor bN \rfloor$ near the middle, that is,*

$x \in \left]\frac{aN}{4}, \frac{3aN}{4}\right[$, *of the rectangle will leave the rectangle through the top boundary is bounded away from zero by* d_3*:*

$$P(\{S^{(x,\lfloor bN \rfloor)} \textit{ hits } \mathfrak{T} \textit{ before } \mathfrak{R} \cup \mathfrak{L} \cup \mathfrak{B}\}) \geq d_3.$$

Furthermore, if we restrict the "starting height"-parameter b above such that for some constant $b_1 > 0$, $0 < b_1 < b < 1$*, then we can extend the above bound* $d_3(a, b)$ *to be uniform in b in the range, id est, there is* $d_3' = d_3'(a) > 0$ *such that the above claim holds for all* $1 \geq b > b_1$.

Proof. Let us now define an auxiliary function $h(x, y) = \sin\left(\pi \frac{1}{aN} x\right) \frac{s^y - s^{-y}}{s^N - s^{-N}}$, where $s = 2 - \cos\left(\frac{\pi}{aN}\right) + \sqrt{\left(2 - \cos\left(\frac{\pi}{aN}\right)\right)^2 - 1}$. The reason we are interested in h is that it is discrete harmonic and on the boundaries it satisfies: $h(x, y) = 0$ in $\mathfrak{R} \cup \mathfrak{L} \cup \mathfrak{B}$, and $0 \leq h(x, y) \leq 1$ in $\mathfrak{T}$.

Let us now show how we found the exact formula for the function h above, that is, how we were able to come up with an ansatz like h. For this purpose, let us study the function A from the proof of the Lemma B.1 more closely. As mentioned above in (B.1) and in connection to it, A is a discrete harmonic function defined on the rectangle R, and in the boundary A satisfies $A(x, y) = 0$ on the right, left and bottom boundaries, and $A(x, y) = 1$ on the top boundary. We shall try to solve the function $A(x, y)$ for all $(x, y) \in V(R)$; we have a discrete Dirichlet boundary value problem for A. We use a method of standard analysis called *separation of variables* (some of the following general discrete analysis material is from [63]); we assume that $A(x, y)$ is of the form $A(x, y) = \psi(x)\phi(y)$. Then we get from the equation (B.1)

$$\frac{1}{4}\left(\psi(x)\phi(y+1) + \psi(x-1)\phi(y) + \psi(x)\phi(y-1) + \psi(x+1)\phi(y)\right) - \psi(x)\phi(y) = 0.$$

Assuming $\psi(x) \geq 0 \leq \phi(y)$ for all x,y, we get from above:

$$\begin{aligned} &\left(\psi(x-1) + \psi(x+1)\right)\phi(y) + \psi(x)\left(\phi(y+1) + \phi(y-1)\right) = 4\psi(x)\phi(y) \\ &\Leftrightarrow \frac{\psi(x-1) + \psi(x+1)}{\psi(x)} = 4 - \frac{\phi(y+1) + \phi(y-1)}{\phi(y)}, \end{aligned}$$

thus yielding that there must exist a constant $\lambda \in \mathbb{R}$, or actually $4 > \lambda > 0$, since $\psi(x) \geq 0 \leq \phi(y)$ (we can choose ψ and ϕ this way, because if for some x_0, $\psi(x_0) = e^{i\alpha}\rho_{x_0}$ for some $\alpha > 0 < \rho_{x_0}$, then because $A(x, y) = \psi(x)\phi(y) \geq 0$, it must be the case that for all y, $\phi(y) = e^{-i\alpha}\sigma_y$ for some $\sigma_y > 0$ so that indeed we must have for all x that $\psi(x) = e^{i\alpha}\rho_x$, so that the complex factor can be eliminated from ψ and ϕ), such that

$$\begin{cases} \lambda = \dfrac{\psi(x-1) + \psi(x+1)}{\psi(x)} > 0, \\ \lambda = 4 - \dfrac{\phi(y+1) + \phi(y-1)}{\phi(y)} < 4. \end{cases}$$

Now let us study the ψ equation first. The equation $\psi(x-1) + \psi(x+1) = \lambda\psi(x)$ is a *second order linear homogeneous constant coefficient difference*

equation; the solution to it is found, in analogy with ordinary second order constant coefficient linear differential equation, to have the form $\psi(x) = r^x$; let us solve for r:

$$\begin{aligned}
&\psi(x-1) + \psi(x+1) - \lambda\psi(x) = r^x\left(r^{-1} + r - \lambda\right) = 0 \\
&\Leftrightarrow r^2 - \lambda r + 1 = 0 \\
&\Leftrightarrow r_\pm = \frac{1}{2}\lambda \pm \sqrt{\frac{1}{4}\lambda^2 - 1}.
\end{aligned}$$

Now depending on the value of λ, we have three cases: firstly $r_\pm \in \mathbb{R}$ and $r_+ > r_-$ for $\lambda > 2$, secondly $r_+ = r_- \in \mathbb{R}$ for $\lambda = 2$ and lastly $r_+ = \overline{r_-} \in \mathbb{C}\setminus\mathbb{R}$, where the line over a variable means complex conjugate, for $\lambda < 2$.

Now we put further demands on our function ψ; because $A(x, y) = 0$ in the right, left and bottom boundary, it follows naturally that we should require that ψ satisfies the boundary condition $\psi(0) = 0$ and $\psi(aN) = 0$. Now let us look on the three cases presented above individually:

$2 < \lambda < 4$: The general solution is

$$\psi(x) = D_1 r_+^x + D_2 r_-^x,$$

as can be seen easily when we note that the solution space to the given linear problem forms a 2-dimensional linear space, because we can define in the space of real number sequences the scalar multiplication by real scalars and summation by the point-wise operations, thus making the space a vector space; then we note that our solution to the given second order difference equation becomes totally fixed by the equation if we fix two members, say, $\psi(0)$ and $\psi(1)$ of the sequence; this means that all solutions can be expressed as linear combination of two solutions ψ_1, ψ_2 that satisfy $\psi_1(0) = 1, \psi_1(1) = 0$ and $\psi_2(0) = 0, \psi_2(1) = 1$ (for if ψ_3 is a solution, then $\psi_3(0)\psi_1 + \psi_3(1)\psi_2$ is a solution having the same first and second value as ψ_3, so that it actually agrees completely with ψ_3: $\psi_3 = \psi_3(0)\psi_1 + \psi_3(1)\psi_2$). Since ψ_1 and ψ_2 are obviously linearly independent, we have that the solution space is two-dimensional.

Now considering our boundary condition we get:

$$\begin{cases} \psi(0) = D_1 + D_2 = 0, \\ \psi(aN) = D_1 r_+^{aN} + D_2 r_-^{aN} = 0. \end{cases}$$

which is a group of linear equations for the coefficients D_1 and D_2; because the group of equations is linearly independent (the determinant for the coefficient matrix is $r_-^{aN} - r_+^{aN} \neq 0$), the only solution is the trivial solution $D_1 = D_2 = 0$ yielding the trivial solution $\psi(x) = 0$ for all x, which does not interest us, so we exclude this case.

$\lambda = 2$: The general solution (because the solution space has two dimensions) is:

$$\psi(x) = D_1 r^x + D_2 x r^x$$

where $r = r_+ = r_-$; the fact that $D_2 x r^x$ is a solution can be seen by direct calculation; let $\psi_1(x) = D_2 x r^x$:

$$\psi_1(x-1) + \psi_1(x+1) = D_2\,(x+1)\,r^{x+1} + D_2\,(x-1)\,r^{x-1} = \lambda\psi(x) = \lambda D_2 x r^x$$
$$\Leftrightarrow x\left(r^2 - \lambda r + 1\right) + \left(r^2 - 1\right) = 0$$

where $r^2 - \lambda r + 1 = 0$ by definition of r; also $r^2 - 1 = 0$ because r is a double root, so that the derivative of $r \mapsto r^2 - \lambda r + 1$, that is, $2r - \lambda$ is zero, so that since $\lambda = 2$, we get $r = 1$.

Now our boundary condition yields:

$$\begin{cases} \psi(0) = D_1 = 0 \\ \psi(aN) = D_2 aN r^{aN} = 0 \Rightarrow D_2 = 0 \end{cases}$$

thus giving only the trivial solution, so we abandon this case also.

Now let $r_+ = \overline{r_-}$ be complex, also $0 < \lambda < 2$. Then we find that a general complex valued solution ψ_1 is

$$\psi_1(x) = D_1 r_+^x + D_2 r_-^x$$

for some coefficients $D_1, D_2 \in \mathbb{C}$. Let $r_+ = \overline{r_-} = |r_+|\,(\cos(\theta) + i\sin(\theta))$; we use the de Moivre formula:

$$\begin{aligned}\psi_1(x) &= D_1 r_+^x + D_2 r_-^x = D_1 |r_+|^x\,(\cos(\theta) + i\sin(\theta))^x + D_2 |r_+|^x\,(\cos(\theta) - i\sin(\theta))^x \\ &= D_1 |r_+|^x\,(\cos(x\theta) + i\sin(x\theta)) + D_2 |r_+|^x\,(\cos(x\theta) - i\sin(x\theta)) \\ &= (D_1 + D_2)\,|r_+|^x \cos(x\theta) + i\,(D_1 - D_2)\,|r_+|^x \sin(x\theta).\end{aligned}$$

Now we see ($|r_+|^x \cos(x\theta)$ and $|r_+|^x \sin(x\theta)$ are real-valued solutions; they are also linearly independent since $\cos(x\theta)$ and $\sin(x\theta)$ are linearly independent, basically because $\cos(x\theta)^2 + \sin(x\theta)^2 = 1$) that any real-valued solution ψ_2 is of the form (one chooses D_1, D_2 above in a suitable way):

$$\psi_2(x) = D_1 |r_+|^x \cos(x\theta) + D_2 |r_+|^x \sin(x\theta)$$

where $D_1, D_2 \in \mathbb{R}$. Now let us calculate $|r_+|$; we have that

$$|r_+| = \left|\frac{1}{2}\lambda + i\sqrt{1 - \frac{1}{4}\lambda^2}\right| = 1$$

so we know that the general solution is actually of the form:

$$\psi(x) = D_1 \cos(\theta x) + D_2 \sin(\theta x)$$

where $\cos(\theta) = \mathrm{Re}(r_+) = \frac{1}{2}\lambda$, $\sin(\theta) = \mathrm{Im}(r_+) = \sqrt{1 - \frac{1}{4}\lambda^2} = \sqrt{1 - \cos(\theta)^2}$, and D_1, D_2 are some constant coefficients. Our boundary condition gives:

$$\begin{cases} \psi(0) = 0 \Leftrightarrow D_1 = 0 \\ \psi(aN) = 0 \Leftrightarrow \theta = \pi\dfrac{k}{aN} \end{cases}$$

for some $k = 1, \ldots, aN - 1$, because we do not want $\psi(x) = 0$ for all x.

Thus we have solved for ψ; let us look for ϕ with $\lambda_k = 2\cos(\theta_k)$ (the separation coefficient corresponding to $\theta_k = \frac{k}{aN}\pi$, $k = 1, \ldots, aN-1$) fixed:

$$\phi_k(y+1) + \phi_k(y-1) = (4 - \lambda_k)\,\phi_k(y).$$

This is again a second order linear constant coefficient difference equation; proceeding as above, let $\phi_k(y) = s(k)^y$, and we get for $s(k)$:

$$s_\pm(k) = 2 - \frac{1}{2}\lambda_k \pm \sqrt{\left(2 - \frac{1}{2}\lambda_k\right)^2 - 1}$$

from where we note that $s_\pm(k) > 0$ ($0 < \lambda < 2$), thus we know that the general solution is of the form:

$$\phi_k(y) = D_3 s_+(k)^y + D_4 s_-(k)^y$$

and from the boundary requirement $A(x,y) = 0$ for (x,y) in the bottom boundary and $A(x,y) = 1$ in the top boundary we get that we should have

$$\phi_k(N) = 1 \Leftrightarrow D_3 s_+(k)^N + D_4 s_-(k)^N = 1$$

and we also want to have

$$\phi_k(0) = 0 \Leftrightarrow D_3 + D_4 = 0$$

so that we get $D_3 = \frac{1}{s_+(k)^N - s_-(k)^N}$, $D_4 = \frac{-1}{s_+(k)^N - s_-(k)^N}$. We note that

$$s_-(k)s_+(k) = 1$$

and thus let us denote $s(k) = s_+(k) = 2 - \frac{1}{2}\lambda_k + \sqrt{\left(2 - \frac{1}{2}\lambda_k\right)^2 - 1}$, $s_-(k) = \frac{1}{s(k)}$. This gives us

$$A(x,y) = \psi(x)\phi(y) = D_2 \sin\left(\pi \frac{k}{aN} x\right) \frac{s(k)^y - s(k)^{-y}}{s(k)^N - s(k)^{-N}}$$

where $k = 1, \ldots, aN-1$.

Now as in the case of the standard separation of variables procedure, we assume that the function A is actually a linear combination of the kind:

$$A(x,y) = \sum_{k=1}^{aN-1} D_k \sin\left(\pi \frac{k}{aN} x\right) \frac{s(k)^y - s(k)^{-y}}{s(k)^N - s(k)^{-N}}.$$

Now we choose h to be the $k = 1$-term of the above series. It is the eigenfunction that corresponds to the largest eigenvalue λ_k of the prescribed discrete Dirichlet problem, and h is also the eigenfunction that has the least oscillation in x-variable, and therefore it is the one we consider relevant.

Now, remembering that $h(x,y) = 0 = A(x,y)$ on the right, left and bottom boundaries, and $0 \le h(x,y) \le 1 = A(x,y)$ on the top boundary, by the maximum principle for discrete harmonic functions presented in appendix Lemma A.1, $h(x,y) \le A(x,y)$ on the whole domain R. Especially

$h(x, \lfloor bN \rfloor) \leq A(x, \lfloor bN \rfloor)$ for $0 < b < 1$ from which we get by choosing $x \in \left[\frac{aN}{4}, \frac{3aN}{4}\right]$ that

$$h(x, \lfloor bN \rfloor) = \sin\left(\pi \frac{1}{aN} x\right) \frac{s^{\lfloor bN \rfloor} - s^{-\lfloor bN \rfloor}}{s^N - s^{-N}} \geq \frac{1}{\sqrt{2}} \frac{s^{\lfloor bN \rfloor} - s^{-\lfloor bN \rfloor}}{s^N - s^{-N}}. \tag{B.2}$$

Hence we get

$$\lim_{N\to\infty} h(x, \lfloor bN \rfloor) \geq \frac{1}{\sqrt{2}} \frac{\exp\left(b\frac{\pi}{a}\right) - \exp\left(-b\frac{\pi}{a}\right)}{\exp\left(\frac{\pi}{a}\right) - \exp\left(-\frac{\pi}{a}\right)} > 0 \tag{B.3}$$

where we note that for all $\alpha \in \mathbb{R}$

$$s^{\alpha N} = \exp\left(\alpha \frac{\ln(s)}{\frac{1}{N}}\right)$$

and that by l'Hôpital's rule (note that $s = s(\theta(N)) = 2 - \cos(\theta(N)) + \sqrt{(2 - \cos(\theta(N)))^2 - 1}$, $\theta(N) = \frac{\pi}{aN}$)

$$\lim_{N\to\infty} \frac{\ln(s)}{\frac{1}{N}} = \lim_{N\to\infty} \frac{\frac{\frac{d}{dN} s(\theta(N))}{s(\theta(N))}}{\frac{-1}{N^2}} = \lim_{N\to\infty} \frac{\frac{s'\left(\frac{\pi}{aN}\right)\cdot\left(\frac{-\pi}{aN^2}\right)}{s\left(\frac{\pi}{aN}\right)}}{\frac{-1}{N^2}} = \frac{\pi}{a} \lim_{N\to\infty} \frac{s'\left(\frac{\pi}{aN}\right)}{s\left(\frac{\pi}{aN}\right)}$$

and we have $s(0) = 1$ (and s is continuous) and

$$\begin{aligned}
\lim_{N\to\infty} s'\left(\frac{\pi}{aN}\right) &= \lim_{\theta\to 0+} s'(\theta) \\
&= \lim_{\theta\to 0+} \left(\sin(\theta) + \frac{1}{2} \frac{1}{\sqrt{(2 - \cos(\theta))^2 - 1}} \cdot 2\,(2 - \cos(\theta)) \sin(\theta)\right) \\
&= 0 + (2 - 1) \cdot \lim_{\theta\to 0+} \frac{\sin(\theta)}{\sqrt{(2 - \cos(\theta))^2 - 1}} \\
&= \sqrt{\lim_{\theta\to 0+} \frac{\sin(\theta)^2}{(2 - \cos(\theta))^2 - 1}}
\end{aligned}$$

and applying l'Hôpital's rule once again

$$\begin{aligned}
\lim_{\theta\to 0+} \frac{\sin(\theta)^2}{(2 - \cos(\theta))^2 - 1} &= \lim_{\theta\to 0+} \frac{\frac{d}{d\theta} \sin(\theta)^2}{\frac{d}{d\theta}\left((2 - \cos(\theta))^2 - 1\right)} \\
&= \lim_{\theta\to 0+} \frac{2 \sin(\theta) \cos(\theta)}{2\,(2 - \cos(\theta)) \sin(\theta)} = 1.
\end{aligned}$$

Putting all together we get

$$\lim_{N\to\infty} s^{\alpha N} = \exp\left(\alpha \frac{\pi}{a} \cdot \frac{\sqrt{1}}{1}\right) = \exp\left(\alpha \frac{\pi}{a}\right).$$

This means that for sufficiently large N, the probability that a random walker that starts at height $\lfloor bN \rfloor$ near the middle ($x \in \left]\frac{aN}{4}, \frac{3aN}{4}\right[$) of the rectangle will leave the rectangle through the top boundary is bounded away

from zero uniformly in N,though the aspect ratio a of the rectangle can affect this uniform in N bound. Since there is only a finite amount of discrete rectangles with given aspect ratio whose height is less than $N_0 \in \mathbb{N}$, this means that actually we can drop the "sufficiently large"-statement above (there obviously is a positive probability for the random walker to exit through the top in any rectangle; since there is only a finite amount of rectangles that we cannot bound by the above limit, we can just take the smallest probability among those rectangles that we cannot bound) and just say that for all N, the probability of exiting at the top for a random walker started at the height $\lfloor bN \rfloor$ near the middle is bounded by a positive constant $d_3(a, b) > 0$ uniformly in N.

Furthermore, we can also see how to attain the stronger result mentioned in the formulation of this Lemma: namely, when we restrict the range of the "starting height"-parameter b as above such that for $b_1 > 0$, $1 \leq b > b_1$, we can see that the above statement of a positive lower bound for the exiting probability holds for all b in the given range by just observing that the lower bound (B.2) can be extended to cover all b in range, id est,

$$h(x, \lfloor bN \rfloor) \geq \frac{1}{\sqrt{2}} \frac{s^{\lfloor b_1 N \rfloor} - s^{-\lfloor b_1 N \rfloor}}{s^N - s^{-N}}$$

and then we can take the limit as in (B.3) and then use the argument that there are only a finite amount of rectangles not bound by the limit, so we can take the minimum of exit probabilities among those rectangles. □

Now let us prove another lemma that is similar to the above. Namely, let us study the probability that a random walker, starting in the middle of the rectangle R exits the rectangle at the middle of the top of the rectangle; again the more formal statement will be apparent from below. We shall show that this probability is bounded below by $\frac{d_4}{N}$ for some $d_4 > 0$.

Lemma B.3. *There exist positive constant d_4 such that for all $N \in \mathbb{N}$, we have that when $Q = (\lfloor \frac{aN}{2} \rfloor, N)$ is the middle point of the top side of the rectangle, then*

$$P(\{S^{(\lfloor \frac{aN}{2} \rfloor, \lfloor \frac{N}{2} \rfloor)} \textit{ hits } Q \textit{ before } \partial R \setminus \{Q\}\}) \geq \frac{d_4}{N}.$$

Proof. So let $Q = (\lfloor \frac{aN}{2} \rfloor, N)$ be the "target point" at the middle of the top of the rectangle. Let us introduce yet another auxiliary function

$$\tilde{A}(x, y) = P(\{S^{(x,y)} \text{ hits } Q \text{ before } \partial R \setminus \{Q\}\}),$$

which is again discrete harmonic on the interior of the rectangle, and has boundary values 0 on the right, left and bottom boundary, and 0 everywhere else on the top boundary except at point Q, in which the boundary value is 1. Now we would like to solve $\tilde{A}$-function explicitly; we have again a discrete Dirichlet boundary value problem, this time for $\tilde{A}$. We shall solve this boundary value problem the same way we did above in Lemma B.2 using the separation of variables technique; let A be again the function from

the proofs of the Lemmas B.1, B.2, and let us use the same denotations as in proof of Lemma B.2 (in a sense, the present Lemma continues from where Lemma B.2 left off, so it is useful for the reader to consider this Lemma as a sequel to the above Lemma B.2). Since $\tilde{A}$ and A are both discrete harmonic and have similar boundary values on the right, left and bottom boundary, we see that most of the procedure that we performed in search of A also applies now. The only case that a priori demands some further consideration is when considering the top boundary value for the separation function ϕ, since A and $\tilde{A}$ have different boundary values at the top boundary. However, we note that since we do not want $\tilde{A}$ to vanish identically on the top boundary, we should actually make the same choice of the top boundary value for ϕ, that is, we demand $\phi(N) = 1$; note that this choice is basically the only sensibly one, since all other choices for $\phi(N) \neq 0$ are basically just scaled versions of the choice $\phi(N) = 1$ (except of course the choice $\phi(N) = 0$, which is a choice we do not want to make, since it would lead to $\tilde{A}$ vanishing identically). So by the above, we get the following representation for $\tilde{A}$:

$$\tilde{A}(x, y) = \sum_{k=1}^{aN-1} D_k \sin\left(\pi \frac{k}{aN} x\right) \frac{s(k)^y - s(k)^{-y}}{s(k)^N - s(k)^{-N}}.$$

Now we should determine the constants D_k, $k = 1, \ldots, aN - 1$ such that the top boundary condition of $\tilde{A}$, that is, attaining value 1 at point Q of the top boundary and 0 elsewhere on the top boundary, is satisfied (as the reader surely notices, thus far we have made no difference between our of the functions A and $\tilde{A}$, but now the determination of these constants is what separates A and $\tilde{A}$; we did not attempt this determination for A as it was unnecessary for our purposes). So far we have not determined the value of our ansatz-representation for $\tilde{A}$ on the top boundary at $y = N$ (we only set $\phi_k(N) = 1$, but this of course restricts the function in no way); now let us do this. We note that real valued functions defined on the set $\{(x, N) \,|\, x = 1, \ldots, aN - 1\}$ form a vector space through pointwise addition and multiplication by real coefficients; in this vector space, which we shall denote by V, we can define an inner product

$$\langle h_1, h_2 \rangle = \sum_{x=1}^{aN-1} h_1(x) h_2(x)$$

where we denote $h_1(x, N) = h_1(x)$ for $h_1, h_2 \in V$. Now let us show that the functions $f_k(x) = \sqrt{\frac{2}{aN}} \sin\left(\pi \frac{k}{aN} x\right)$ form an orthonormal basis of V with respect to this inner product. So let $\theta = \frac{k}{aN}\pi$, $\theta' = \frac{k'}{aN}\pi$ for some $k, k' \in \{1, \ldots, aN - 1\}$. Then let us define $q = e^{i(\theta - \theta')}$, $p = e^{i(\theta + \theta')}$. Then $p \neq 1$, and $q \neq 1 \Leftrightarrow k \neq k'$. Let $k \neq k'$ and let us calculate using the geometric sum

formula:

$$\begin{aligned}
&\sum_{x=1}^{aN-1} \sin(\theta x)\sin(\theta' x) = \sum_{x=1}^{aN-1} \frac{1}{-4}\left(e^{i\theta x} - e^{-i\theta x}\right)\left(e^{i\theta' x} - e^{-i\theta' x}\right) \\
&= \sum_{x=0}^{aN-1} \frac{1}{4}\left(q^x + q^{-x} - p^x - p^{-x}\right) \\
&= \frac{1}{4}\left(\frac{1-q^{aN}}{1-q} + \frac{1-q^{-aN}}{1-\frac{1}{q}} - \frac{1-p^{aN}}{1-p} - \frac{1-p^{-aN}}{1-\frac{1}{p}}\right) \\
&= \frac{1}{4}\left(\frac{1-q^{aN}-q+q^{-aN+1}}{1-q} - \frac{1-p^{aN}-p+p^{-aN+1}}{1-p}\right) \\
&= \frac{1}{4}\left(\frac{q^{aN}-q^{2aN}-q^{aN+1}+q}{(1-q)\,q^{aN}} - \frac{p^{aN}-p^{2aN}-p^{aN+1}+p}{(1-p)\,p^{aN}}\right) \\
&= \frac{1}{4}\left(\frac{q^m(1-q)-(1-q)}{(1-q)\,q^m} - \frac{p^m(1-p)-(1-p)}{(1-p)\,p^m}\right) \\
&= \frac{1}{4}\left(1 - \frac{1}{q^m} - 1 + \frac{1}{p^m}\right) \\
&= \frac{1}{4}\left(p^{aN} - q^{aN}\right) \\
&= \frac{1}{4}\left(e^{i(k+k')\pi} - e^{i(k-k')\pi}\right) \\
&= \frac{i}{2}e^{ik\pi}\sin(k'\pi) = 0
\end{aligned}$$

where we note $q^{2aN} = 1 = p^{2aN}$, so $q^{aN} = \frac{1}{q^{aN}}$, $p^{aN} = \frac{1}{p^{aN}}$.

In the case $k = k'$ we get $q = 1$ and

$$\begin{aligned}
\sum_{x=1}^{aN-1} \sin(\theta x)\sin(\theta' x) &= \sum_{x=0}^{aN-1} \frac{1}{4}\left(q^x + q^{-x} - p^x - p^{-x}\right) \\
&= \frac{1}{4}\left(2aN - \sum_{x=0}^{aN-1} p^x + p^{-x}\right) \\
&= \frac{1}{2}aN - \frac{1}{4}\frac{1-p^{aN}-p+p^{-aN+1}}{1-p} \\
&= \frac{1}{2}aN - \frac{1}{4}\left(1 - \frac{1}{p^{aN}}\right) \\
&= \frac{1}{2}aN - \frac{1}{4}\left(1 - e^{i(k+k')\pi}\right) \\
&= \frac{1}{2}aN - \frac{1}{4}\left(1 - e^{i2k\pi}\right) = \frac{1}{2}aN
\end{aligned}$$

from which we can deduce that the functions $\sqrt{\frac{2}{aN}}\sin\left(\pi\frac{k}{aN}x\right)$ form an orthonormal basis as claimed.

Now we get

$$\tilde{A}(x,N) = \sum_{k=1}^{aN-1} D_k \sin\left(\pi \frac{k}{aN} x\right) = \delta_{x,\lfloor \frac{aN}{2} \rfloor}$$

where $\delta_{a,b}$ is the Kronecker delta; hence our coefficients D_k should be:

$$D_k = \left\langle \sqrt{\frac{2}{aN}} \delta_{x,\lfloor \frac{aN}{2} \rfloor}, \sqrt{\frac{2}{aN}} \sin\left(\pi \frac{k}{aN} x\right) \right\rangle = \frac{2}{aN} \sin\left(\pi \frac{k}{aN} \left\lfloor \frac{aN}{2} \right\rfloor\right).$$

We arrive at the solution

$$\tilde{A}(x,y) = \sum_{k=1}^{aN-1} \frac{2}{aN} \sin\left(\pi \frac{k}{aN} \left\lfloor \frac{aN}{2} \right\rfloor\right) \sin\left(\pi \frac{k}{aN} x\right) \frac{s(k)^y - s(k)^{-y}}{s(k)^N - s(k)^{-N}}.$$

Now the quantity we are interested is

$$\tilde{A}\left(\left\lfloor \frac{aN}{2} \right\rfloor, \left\lfloor \frac{N}{2} \right\rfloor\right) = \sum_{k=1}^{aN-1} \frac{2}{aN} \sin\left(\pi \frac{k}{aN} \left\lfloor \frac{aN}{2} \right\rfloor\right)^2 \frac{s(k)^{\lfloor \frac{N}{2} \rfloor} - s(k)^{-\lfloor \frac{N}{2} \rfloor}}{s(k)^N - s(k)^{-N}}.$$

Let us approximate this value as follows ($s(k) \geq 1$ for all $k = 1, \ldots, aN-1$):

$$\begin{aligned}
\tilde{A}\left(\left\lfloor \frac{aN}{2} \right\rfloor, \left\lfloor \frac{N}{2} \right\rfloor\right) &\geq \frac{2}{aN} \sin\left(\pi \frac{1}{aN} \left\lfloor \frac{aN}{2} \right\rfloor\right)^2 \frac{s(1)^{\lfloor \frac{N}{2} \rfloor} - s(1)^{-\lfloor \frac{N}{2} \rfloor}}{s(1)^N - s(1)^{-N}} \\
&\geq \frac{2}{aN} \left(\frac{1}{\sqrt{2}}\right)^2 \frac{s(1)^{\lfloor \frac{N}{2} \rfloor} - s(1)^{-\lfloor \frac{N}{2} \rfloor}}{s(1)^N - s(1)^{-N}} \\
&\geq \frac{1}{aN} \frac{s(1)^{\frac{1}{4}N} - s(1)^{-\frac{1}{4}N}}{s(1)^N - s(1)^{-N}}.
\end{aligned}$$

Hence we get, using the same argumentation as above in Lemma B.2

$$\lim_{N\to\infty} \frac{s(1)^{\frac{1}{4}N} - s(1)^{-\frac{1}{4}N}}{s(1)^N - s(1)^{-N}} = \frac{\exp\left(\frac{1}{4}\frac{\pi}{a}\right) - \exp\left(-\frac{1}{4}\frac{\pi}{a}\right)}{\exp\left(\frac{\pi}{a}\right) - \exp\left(-\frac{\pi}{a}\right)} > 0,$$

thus showing that, for N large enough, there exists a positive constant d_4 such that

$$\tilde{A}\left(\left\lfloor \frac{aN}{2} \right\rfloor, \left\lfloor \frac{N}{2} \right\rfloor\right) \geq \frac{d_4}{N}. \tag{B.4}$$

Now using a similar argument as above about the finite number of discrete rectangles with fixed aspect ratio a and height bounded above by $N_0 \in \mathbb{N}$, since obviously in any rectangle the probability to perform the kind of random walk $\tilde{A}$ postulates is positive, we can see (as we did above with A) that we can relax the "for N large enough"-demand in (B.4). Hence we are done. □

Let us now prove another lemma, so-called *Harnack inequality* (for discrete harmonic functions); we learned this proof from [14]. Note that the fact that the domain R is a rectangle plays no role whatsoever (quite unlike in the previous Lemmas B.1, B.2, B.3, where it was essential) in the following Lemma B.4, but actually any king of simple lattice domain would

suffice in place of R, since the proof of the following Lemma is more topological than geometrical in nature. This tolerance towards the underlying domain is useful in the context where we shall apply the following Lemma B.4.

Lemma B.4. *For any discrete harmonic function $H : V(R) \to [0, \infty[$ there exists constants d_5, $d_6 > 0$ such that for any $u \in V(R)$ and for all v, $w \in B(u, \frac{1}{8}\delta(u, \partial R))$ (δ is the graph metric) we have:*

$$d_5 H(w) \leq H(v) \leq d_6 H(w).$$

Remark: Let us note that the factor $\frac{1}{8}$ has no special role above, and actually any constant $0 < q < 1$ would do instead (the constants d_5, d_6 will of course depend on q); the factor $\frac{1}{8}$ was chosen because it is sufficient for our purposes, and using it allows one to simplify the geometry used in the proof (see figure 69).

Proof. Let $u \in V(R)$ and v, $w \in B(u, \frac{1}{8}\delta(u, \partial R)) = \{x \in V(R) \,|\, \delta(x, u) < \frac{1}{8}\delta(u, \partial R)\}$. Note that if $\delta(u, \partial R) = \inf\{\delta(x, u) \,|\, x \in \partial R\} < 8$, the claim is obviously true, so let $\delta(u, \partial R) \geq 8$ (so that v and w do really exist). We note that by the maximum principle, as demonstrated in the appendix Lemma A.1, there exists a nearest-neighbour path $\gamma(v)$ from v to ∂R such that $H \geq H(v)$ along this path. This is because we can study the connected component C_v to which v belongs of the graph G_v^1, which is the subgraph of R that has as vertex set $\{x \in V(R) \,|\, H(x) \geq H(v)\}$ and that inherits all edges of R between its vertexes. Now, if C_v contains some boundary vertex, the claim is obviously true (because C_v is connected). So let us assume towards a contradiction that C_v contains no boundary vertexes. Let us then form yet another subgraph of R, say G_v^2, that contains G_v^1 and its outer boundary, that is, those vertexes that have a neighbour in G_v^1. Now by the maximum principle, since H is discrete harmonic in G_v^2, it attains its maximum in the boundary of G_v^2; but since the graph G_v^1 contained no boundary vertexes, the boundary of G_v^2 is actually contained in the outer boundary of G_v^1; but in the outer boundary of G_v^1, $H < H(v)$ by the definition of G_v^1; so H cannot attain its maximum at the boundary of G_v^2, a contradiction. So actually C_v must contain a boundary vertex of R. So the path $\gamma(v)$ must exist.

Then we note that there exists a constant d_7 such that the probability that a random walk $\mathbf{S}^z$ in the infinite square lattice (also no boundary effects here) started at point $z \in B(u, \frac{3}{4}r) \setminus B(u, \frac{1}{2}r)$ makes a full turn inside the annulus $B(u, r) \setminus B(u, \frac{1}{8}r)$ and crosses its own trajectory afterwards thus making a *non-trivial circle* in the annulus, denote this event $\mathrm{CA}(u, r, \frac{1}{8}r)$, is bounded away from zero by $d_6 > 0$ uniformly in z, u, and $r > 0$ (again, there is nothing special about the factors $\frac{3}{4}$ and $\frac{1}{2}$, but any factors q_1 and q_2 such that $0 < q < q_1 < q_2 < 1$, where q is the "size-factor" of the inner boundary of the annulus, also $q = \frac{1}{8}$ here, would do; we choose factors $\frac{3}{4}$, $\frac{1}{2}$ for simplicity). Let us prove this by using an argument similar to the "block arguments" we utilized in the study of percolation. Namely, we can make

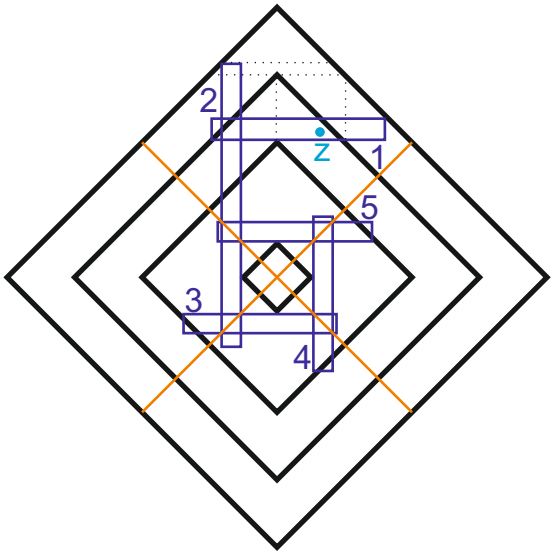


FIGURE 69. The proof of Harnack inequality. The segments separated by the orange lines are in symmetric position, so we may assume we start (from point z) in the upper segment. We define an auxiliary domain by using 5 rectangles positioned like in this figure.

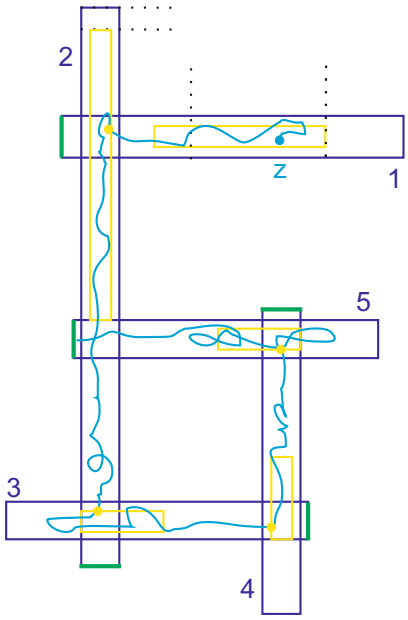


FIGURE 70. The proof of Harnack inequality. The borders of the rectangles to which the random walk must limit itself are in blue, the "starting areas" are in yellow, and the "finish lines" are in green. A tentative random walk is depicted in light blue. Note the black dotted lines, which signal that the starting area of rectangles 1 and 2 can be limited as to not come near the border of the rectangles.

a circle-path in the annulus from 5 rectangles - call these rectangles R_i, $i = 1, \ldots, 5$ - as depicted in figure 69, with rectangle 1 positioned such that the initial point z is in the "starting area" (to be defined below) of that rectangle; note also that because of the four-fold rotational symmetry of the situation, the 4 segments separated by the orange lines in figure 69 of the ball $B(u, r)$ are in symmetric position, so we can assume the situation looks like that of figure 69. We intend to use the above result of Lemma B.2 that the probability for a random walker starting in the middle of the rectangle exiting from the top being bounded below by a positive constant uniformly in the size of the rectangle. Let us therefore define that in each rectangle, the area depicted yellow in the figure 70 is to be called the "starting area", and the green line of the rectangle is the "finish line". As apparent from above, the important thing is the ratios of the starting area (whose width is half the width of the whole rectangle, and that is positioned width-wise in the middle of the whole rectangle) to the whole rectangle and the aspect ratio of the whole rectangle, which we obviously can fix for the 5 rectangles of figure 69 in a way that is independent of r (and depends only from the shape of the annulus $B(u, r) \setminus B(u, \frac{1}{8}r)$ and from the fact that we chose the initial point to lie in the annulus $B(u, \frac{3}{4}r) \setminus B(u, \frac{1}{2}r)$, id est, the only relevant thing is the relation of the constants $\frac{1}{8}$, $\frac{1}{2}$, $\frac{3}{4}$ and 1, 1 being the size-factor of the whole annulus; as noted above, we could have chosen some other ratios). Now the probability of an event SF(i), $i = 1, \ldots, 5$ that a random walker, beginning to walk in the starting area of any rectangle R_i, exits that rectangle (that is, meets the boundary for the first time) through the finish line, is bounded below by a constant depending on the aspect ratio of the rectangle and the relation of the starting area's height to the height of the whole rectangle, but not on the size on the rectangle, and therefore, not on the size of the annulus. But as can be seen from figure 70, a random walker must cross the starting area of the next rectangle on his way to the finish line; let us denote this event that starting from the starting area of rectangle R_i the random walker $\mathbf{S}$ walks inside the rectangle R_i to the starting area of rectangle R_{i+1} by SS(i), $i = 1, \ldots, 4$. Now we can again use result that the probability that a random walker that has entered the starting area of the next rectangle exits the next rectangle through the finish line is bounded below by a positive constant. Note that the simple random walk being executed here has the property that the walk is "time-wise independent", meaning that two different time-sections of the walk that do not overlap are independent. Hence the random walks in the different rectangles (a random walk inside a rectangle begins when the walker enters the starting area of the rectangle, and ends when the walker enters the starting area of the next rectangle) are independent.

So, we arrange our rectangles (see figures 69 and 70) such that the starting point z is in the starting area of the first rectangle, and then we let the random walker walk towards the finish line of the first rectangle and when the walker enters the starting area of the second rectangle, we "change its

direction" to the finish line of the second rectangle. Then we continue inductively: the random walker must cross the starting area of the third rectangle when walking towards the finish line of the second rectangle, and then we study the walk in the third rectangle. Finally, when we reach the fifth rectangle, we let the random walker cross the second rectangle and continue to the finish line of the fifth rectangle, thus ensuring that the walk crosses its own path. The total probability is

$$\begin{aligned} P(\{\mathbf{S}^z \in \mathrm{CA}(u, r, \frac{1}{8}r)\}) &\geq P(\mathrm{SS}(1)) \cdot P(\mathrm{SS}(2)) \cdot P(\mathrm{SS}(3) \cdot P(\mathrm{SS}(4) \cdot P(\mathrm{SF}(5) \\ &\geq P(\mathrm{SF}(1)) \cdot P(\mathrm{SF}(2)) \cdot P(\mathrm{SF}(3))^3 = d_7 > 0. \end{aligned}$$

Note that rectangles 3, 4, 5 are completely similar.

Let us now define a *stopping time*

$$\tau_{\partial R \cup \gamma(v)}(\omega) = \inf\{n \in \mathbb{N} \mid S_n(\omega) \in \partial R \cup \gamma(v)\},$$

where ω is an element in the probability space underlying the random walk S, and ∂R is the set of boundary vertexes of R and $\gamma(v)$ is understood as the set of vertexes of the path $\gamma(v)$; so the stopping time $\tau_{\partial R \cup \gamma(v)}$ is actually a random variable. Then we claim that $H(z) = E_z(H(S_{\tau_{\partial R \cup \gamma(v)}}))$ for $z \in V(R)$, where by the denotation E_z we mean that we study only those random walks that start at z. Let us denote $E(z) = E_z(H(S_{\tau_{\partial R \cup \gamma(v)}}))$, $z \in V(R)$.

So, let $z \in V(R)$. If $z \in \partial R \cup \gamma(v)$, the claim is clear because of the definition of a stopping time (trivially, the expectation of a constant is just this constant itself). So let $z \ni \partial R \cup \gamma(v)$, and let R_z be the connected component of $R \setminus (\partial R \cup \gamma(v))$ to which z belongs; let us abuse the notation and include the outer boundary of R_z, id est, those vertexes of R with graph distance 1 to R_z, in R_z. Note that the aforementioned outer boundary necessarily is a subset of $\partial R \cup \gamma(v)$. Now $R_z \neq \emptyset$ and we note that both H (by assumption) and E are discrete harmonic in the interior of R_z (interior consists of those vertexes having 4 neighbours in R_z); namely, let $y \in R_z$ be an interior point:

$$\begin{aligned} E(y) &= E_y(H(S_{\tau_{\partial R \cup \gamma(v)}})) \\ &= \frac{1}{4}E_{y+i}(H(S_{\tau_{\partial R \cup \gamma(v)}})) + \frac{1}{4}E_{y-1}(H(S_{\tau_{\partial R \cup \gamma(v)}})) + \frac{1}{4}E_{y-i}(H(S_{\tau_{\partial R \cup \gamma(v)}})) \\ &\quad + \frac{1}{4}E_{y+1}(H(S_{\tau_{\partial R \cup \gamma(v)}})) \\ &= \frac{1}{4}\left(E(y+i) + E(y-1) + E(y-i) + E(y+1)\right) \end{aligned}$$

from which the discrete harmonicity at y obviously follows; above we use the probabilistic interpretation of the expectation (the random walker must step to one of the neighbouring vertexes, and it chooses the vertex it step to with equal probability among the neighbours) and we denote the neighbours of y by $y+c$, where $c = i$ for the neighbour above y, $c = -1$ for the neighbour left of y, $c = -i$ for the neighbour below y and $c = -1$ for neighbour right of y.

Now by the definition of the stopping time H and E also have the same boundary values at the boundary $\partial R_z \subset \partial R \cup \gamma(v)$ of our domain (as was noted above). So by the maximum principle of appendix Lemma A.1 applied in R_z to the discrete harmonic function $H - E$ having boundary values 0, we see that $H - E = 0$ on the whole of R_z, and especially on z, also $H(z) = E(z)$. So we have proven the claim that $H(z) = E_z(H(S_{\tau_{\partial R \cup \gamma(v)}}))$ for $z \in V(R)$.

Now, since H is a non-negative function (below we use this by limiting the area of integration and noticing that this must lead to the value of the integral decreasing), we see that for $z \in B(u, \frac{3}{4}\delta(u, \partial R)) \setminus B(u, \frac{1}{2}\delta(u, \partial R))$ (we denote $E(X; A) = \int_A X dP$ for random variable X and an event A),

$$\begin{aligned} H(z) &= E_z(H(S_{\tau_{\partial R \cup \gamma(v)}})) \\ &\geq E_z(H(S_{\tau_{\partial R \cup \gamma(v)}}); S \text{ hits } \gamma(v) \text{ before } \partial R) \\ &\geq E_z(H(S_{\tau_{\partial R \cup \gamma(v)}}); S \in \mathrm{CA}(u, r, \frac{1}{8}r)) \\ &\geq d_7 H(v), \end{aligned}$$

because $H \geq H(v)$ on $\gamma(v)$, and when the random walker makes the above described full turn in the given annulus (the event $\mathrm{CA}(u, r, \frac{1}{8}r)$), it is bound to hit $\gamma(v)$ so that the point in which $H(S_{\tau_{\partial R \cup \gamma(v)}})$ is evaluated necessarily belongs to $\gamma(v)$. Applying the maximum principle for discrete harmonic functions shown in appendix Lemma A.1 we get $H(w) \geq d_7 H(v)$, for the function H must get its minimum in $\overline{B(u, \frac{3}{4}\delta(u, \partial R))} = \{x \in V(R) \,|\, \delta(x, \partial R) \leq \frac{3}{4}\delta(u, \partial R)\}$ on the circle $\{x \in V(R) \,|\, \delta(x, \partial R) = \frac{3}{4}\delta(u, \partial R)\}$, and in this circle the above estimate holds. By reversing the roles of v and w we get $H(v) \geq d_7 H(w)$; also we can choose $d_5 = d_7 > 0$, $d_6 = \frac{1}{d_7} > 0$ as constants we wanted. □

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